

## Representations of the symmetric group

### Homework 4

(Due on 13/02/2015 at 9:10 a.m.)

#### Instructions:

- Solutions must be complete and legible in order to earn maximum points.
- You may discuss and work together if necessary but you must write your own solutions.

1. Let  $N$  be a normal subgroup of  $G$  and let  $\rho : G/N \rightarrow \text{GL}(V)$  be a representation of  $G/N$ . We showed in class that this representation can be lifted to a representation of  $\psi$  of  $G$ . Prove that:

- (a) If  $\rho$  is faithful then kernel of  $\psi$  is  $N$ .
- (b)  $\rho$  is irreducible iff  $\psi$  is.

2. This example shows that the process in the above example can be reversed. Let  $\psi : G \rightarrow \text{GL}(V)$  be a representation of  $G$  and let  $N$  be a normal subgroup of  $G$  contained in the kernel of  $\psi$ . Define  $\rho : G/N \rightarrow \text{GL}(V)$  by  $\rho(gN) = \psi(g)$ .

Show that  $\rho$  is a well-defined representation of  $G/N$ . For what  $N$  will  $\rho$  be faithful?

3. Let  $\rho : G \rightarrow \text{GL}(V)$  be a representation of  $G$  and let  $N$  be the kernel of  $\rho$ .

- (a) Suppose  $\rho$  has character  $\chi$  and degree  $d$ . Prove that  $g \in N$  iff  $\chi(g) = d$ .
- (b) Let  $H$  be a subgroup of  $G$  and consider the corresponding coset representation of  $G$ . Show that in this case,  $N = \cap_i g_i H g_i^{-1}$ , where  $\{g_i\}_{i=1}^k$  is a transversal for  $H$  in  $G$ .

4. Find the conditions under which each of the following representations are faithful: trivial, regular, coset, sign (for  $G = S_n$ ), defining (for  $G = S_n$ ) and degree 1 for  $C_n$ , the cyclic group of order  $n$ .

(The defining representation of  $S_n$  is the degree  $n$  permutation representation of  $S_n$ , arising from the permutation action on the coordinates of a point in  $\mathbb{C}^n$ .)