23/01/2015

Representations of the symmetric group

Homework 3

(Due on 30/01/2015 at 9:10 a.m.)

Instructions:

- Solutions must be complete and legible in order to earn maximum points.
- You may discuss and work together if necessary but you must write your own solutions. Copied solutions (from each other or books or the internet) are easy to identify and easier to grade as they can only earn a zero.

1. Let V be the standard representation of S_3. Decompose the following representations as a direct sum of irreducibles.
   (a) $V \otimes V$
   (b) $\mathrm{Sym}^2 V$
   (c) $\bigwedge^2 V$

2. The symmetric group S_4 acts on the unit cube by permuting the 4 long diagonals of the cube. Consequently, S_4 acts on the faces, vertices and edges of the cube, thus giving rise to a representation associated with each action.

   Let F (resp. E, V) be the representation associated with the action of S_4 on the faces (resp. edges, vertices) of the cube. Decompose F, E and V as a direct sum of irreducible representations.

   Hint: A cube has 6 faces, so dim $F = 6$. Label each face and analyze the action of each conjugacy class of S_4 on the faces. For example, the transposition $\sigma = (1\ 2)$ acts as rotation of the cube by $180°$ about the axis connecting the midpoints of a pair of opposite edges. Observe that no faces remain fixed under this action, so that $\chi_F(\sigma) = 0$. Similar observations will tell you that $(1\ 2\ 3)$ leaves no faces fixed, while $(1\ 2\ 3\ 4)$ fixes 2 faces. Use this information to figure out χ_F.

3. Suppose g is an element of order 2 in G and $\rho : G \to \mathrm{GL}(V)$ is a degree 2 representation of G. If χ_V is known then find the eigenvalues of each ρ_g. Do the same if order if g is $3, 4$ and dimension of V is $3, 4$.

4. Let $G = D_8$ be the dihedral group (group of symmetries of a square).
   (a) Find the conjugacy classes of G.
   (b) Find all irreducible representations of G.
   (c) Form the character table for G.