06/01/2015

Representation theory

Homework 1

(Due on 13/01/2015 at 2:00 p.m.)

Instructions:

- Solutions must be complete and legible in order to earn maximum points.
- You may discuss and work together if necessary but you must **write your own solutions**. Copied solutions (from each other or books or the internet) are easy to identify and easier to grade as they can only earn a zero.

1. Let V be a finite-dimensional vector space and W be a subspace of V. Prove that there is a bijective correspondence between the projections of V onto W and the complements of W in V.

2. Let V be a representation of G containing a vector v such that the set $\{\rho_s(v)\}_{s\in G}$ forms a basis of V. Prove that V is isomorphic to the regular representation of G.

   (Here $\rho : G \to \mathrm{GL}(V)$ is the representation of G in V and ρ_s stands for the element $\rho(s) \in \mathrm{GL}(V)$.)

3. Let V be the regular representation of G and W be the 1-dimensional subspace of V generated by the element $x = \sum_{s\in G} e_s$. Prove that W is a subrepresentation of V and that W is isomorphic to the trivial representation of G.

4. Consider the symmetric group S_n consisting of all bijections of the set $\{1, 2, \ldots, n\}$ onto itself. The elements of S_n are called *permutations*. Every element of S_n can be written as a product of disjoint cycles. The *cycle type* of an element $\pi \in S_n$ is defined to be an expression of the form $(1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ where m_k is the number of cycles of length k in π.

   Prove that

   (a) Two permutations in S_n lie in the same conjugacy class if and only if they have the same cycle type.

   (b) There is a bijection between the conjugacy classes of S_n and the partitions of n.