# SCHUTZENBERGER'S JEU DE TAQUIN

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Abstract. In this article we give a description of Sliding process and Jeu De Taquin which gives an alternate description of the Knuth equivalence of two skew tableaux.

## 1. Introduction

There are two fundamental operations on tableaux from which most of their combinatorial properties can be deduced: the Schensted bumping algorithm and the Schützenberger sliding algorithm. When repeated, the Schensted algorithm leads to the Robinson Schensted Knuth correspondence and the sliding algorithm leads to JEU DE TAQUIN. We aim to show that equivalence of two tableaux (such that one is obtained from the other by some sequence of slides) is same as the Knuth equivalence of those two tableaux.

## 2. Row Insertion and Column Insertion

In this section we will discuss the insertion process and the preliminary definitions and lemmas associated to it.

**Definition 2.1.** *Let $\mu \subseteq \lambda$ as Ferrers diagram. Then $\lambda/\mu$ is called the corresponding* skew diagram *or* skew shape. *It is defined as follows : $\lambda/\mu = \{c : c \in \lambda \text{ and } c \notin \mu\}$. A skew diagram is normal if $\mu = 0$.*

**Definition 2.2.** *A* partial tableau *is an array with distinct entries whose rows and columns increase. (So a partial tableau will be standard if its elements are precisely $1, 2, \ldots, n$.)*

**Definition 2.3.** *If $\lambda$ is a diagram, then an* inner corner *of $\lambda$ is a node $(i, j) \in \lambda$ whose removal leaves the Ferrers diagram of a partition.*

**Definition 2.4.** *If $\lambda$ is a diagram, then an* outer corner *of $\lambda$ is a node $(i, j) \notin \lambda$ whose addition produces the Ferrers diagram of a partition.*

**Definition 2.5.** *Let $P$ be a partial tableau. $x$ is an element not in $P$.* Row inserting *$x$ in $P$ means*

(1) *Set $R = 1st$ row of $P$.*
(2) *While $x$ is less than some element of row $R$, do*
   (i) *Let $y$ be the smallest element of $R$ greater than $x$ and replace $y$ by $x$ in $R$ (denoted by $R \leftarrow x$).*
   (ii) *Set $x := y$ and $R :=$ the next row down*
(3) *Now $x$ is greater than every element of $R$, so place $x$ at the end of row $R$ and stop.*

*Let the new tableau formed after row insertion of $x$ in $P$ be $P'$. Then we write $r_x(P) = P'$ .*

**Definition 2.6.** *Suppose that $Q$ is a partial tableau of shape $\mu$ and that $(i, j)$ is an outer corner of $\mu$. If $k$ is greater than every element of $Q$, then to* place *$k$ in $Q$ at cell $(i, j)$, merely set $Q_{i,j} := k$. The restriction on $k$ guarantees that the new array is still a partial tableau.*

**Definition 2.7.** *With every permutation $\pi \in S_n$, we can associate two tableaux $P$ an $Q$ called the* P tableau *or the* insertion tableau *and* Q tableau *or the* recording tableau *respectively. Their construction can be done in the following way :*

*Let* $\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \end{pmatrix}$.

*We construct a sequence of tableau pairs :*
$(P_o, Q_o) == (0,0), (P_1, Q_1), (P_2, Q_2), ..., (P_n, Q_n) == (P, Q)$
*where* $x_l, x_2, ..., x_n$ *are inserted into the* $P$*'s and* $1, 2, ..., n$ *are placed in the* $Q$*'s so that shape* $P_k =$ *shape* $Q_k$ *for all* $k$
*i.e. starting with the pair of empty tableaux* $(P_o, Q_o)$ *and assuming that* $(P_{k-1}, Q_{k-1})$ *is already constructed, we define* $(P_k, Q_k)$ *as*
$P_k = r_{x_k}(P_{k-1})$
$Q_k =$ *tableau formed by the placement of* $k$ *at box* $(i, j)$ *of* $Q_{k-1}$ *where* $(i, j)$ *is also the box of* $P_k$ *where insertion has terminated.*
$P = P_n$ *and* $Q = Q_n$

(Note: There is in fact a bijection between permutations in $S_n$ and pairs of standard tableaux of shape $\lambda$ where $\lambda$ is a partition of $n$. Hence given two standard tableaux $P$ and $Q$ , we can in fact get back the permutation $\pi$ associated to it)

**Definition 2.8.** Column insertion *is same as row insertion. Just in place of rows we work with columns. Column inserting* $x$ *in a tableau* $P$ *to obtain tableau* $P'$ *is denoted as* $c_x(P) = P'$.

**Lemma 2.9.** *Let* $P$ *be a partial tableau.* $x$ *is an element not in* $P$*. Suppose during the insertion* $r_x(P) = P'$*, the elements* $x', x", \ldots$ *are bumped off from boxes* $(1, j'), (2, j"), \ldots$ *respectively. Then*

(1) $x < x' < x" < \ldots$
(2) $j' \geq j" \geq \ldots$
(3) $P'_{i,j} \leq P_{i,j}$

**Proposition 2.10.** *$P$ is a partial tableau. $x$ and $y$ are distinct elements not in $P$ . Then*

$$c_y r_x(P) = r_x c_y(P)$$

**Theorem 2.11.** *If* $P(\pi) = P$*, then* $P(\pi^r) = P^t$ *where* $t$ *denotes transposition. (If* $\pi = x_1 x_2 \ldots x_n$ *then* $\pi^r$ *is the reversal of* $\pi$*, i.e.,* $\pi^r = x_n x_{n-1} \ldots x_1$*.)*

*Proof.*

$$\begin{aligned} P(\pi^r) = r_{x_1}...r_{x_{n-1}} r_{x_n}(\phi) \quad &\Rightarrow \quad P(\pi^r) = r_{x_1}...r_{x_{n-1}} c_{x_n}(\phi) \text{ (Initially tableaux is empty)} \\ &\Rightarrow \quad P(\pi^r) = c_{x_n} r_{x_1}...r_{x_{n-1}}(\phi) \text{ (by Proposition 2.10)} \\ &\Rightarrow \quad P(\pi^r) = c_{x_n} r_{x_1}...c_{x_{n-1}}(\phi) \\ &\vdots \\ &\Rightarrow \quad P(\pi^r) = c_{x_n} c_{x_{n-1}}...c_{x_1}(\phi) \text{ (By induction)} \\ &\Rightarrow \quad P(\pi^r) = P^t \text{ (By definition of column insertion)} \end{aligned}$$

□

**Definition 2.12.** *Two permutations* $\pi$ *and* $\sigma \in S_n$ *are* P-equivalent *i.e.* $\pi \overset{P}{\cong} \sigma$*, if* $P(\pi) = P(\sigma)$*. For example.* $\pi =$*736254 and* $\sigma =$*732654 are P-equivalent.*

## 3. Knuth Equivalence

In this section a description of Knuth relations is given and the theorems associated to it are explained.

**Definition 3.1.** *Suppose* $x < y < z$*. Then* $\pi$ *,*$\sigma \in S_n$ *differ by a* Knuth relation of the 1st kind *i.e.* $\pi \overset{1}{\cong} \sigma$ *if*
$\pi = x_1 \ldots . yxz \ldots x_n$ *and* $\sigma = x_1 \ldots yzx \ldots x_n$ *or vice versa.*
*They differ by a* Knuth relation of 2nd kind *i.e.* $\pi \overset{2}{\cong} \sigma$ *if*
$\pi = x_1 \ldots xzy \ldots x_n$ *and* $\sigma = x_1 \ldots zxy \ldots x_n$ *or vice versa.*

**Definition 3.2.** *Two permutations* $\pi$ ,$\sigma \in S_n$ *are* Knuth equivalent *i.e.* $\pi \overset{K}{\cong} \sigma$ , *if there is a sequence of permutations such that*
$\pi = \pi_1 \overset{i}{\cong} \pi_2 \overset{j}{\cong} \cdots \overset{l}{\cong} \pi_k = \sigma$ *where* $i, j, \dots l \in \{1, 2\}$

*for example.* $\pi = 736254$ *and* $\sigma = 732654$ *are Knuth-equivalent as they differ by a Knuth relation of the 1st kind.*

**Definition 3.3.** *Let* $P$ *be a tableau.* Row word *of* $P$ *is a permutation*

$\pi_P = R_l R_{l-1} ... R_1$ *where* $R_i$ *(*$i = 1, 2, ...l$*) is the row of* $P$*.*

**Lemma 3.4.** *If* $P$ *is a standard tableau, then insertion tableau of* $\pi_P$ *is* $P$*.*

**Theorem 3.5.** *If* $\pi$ , $\sigma \in S_n$*, then*

$$\pi \overset{K}{\cong} \sigma \Longleftrightarrow \pi \overset{P}{\cong} \sigma$$

**Definition 3.6.** *Two skew partial tableaux* $P$ *and* $Q$ *are said to be Knuth equivalent*
*i.e.* $P \overset{K}{\cong} Q$ *if* $\pi_p \overset{K}{\cong} \pi_Q$.

## 4. Sliding and Jeu de Taquin

**Definition 4.1.** *Let* $P$ *be a partial tableau of shape* $\lambda/\mu$*. A* forward slide *from box* $c$ *is performed as follows :*

(1) *Pick* $c$ *to be an inner corner of* $\mu$
(2) *While* $c$ *is not an inner corner of* $\lambda$*, do*
(i) *If* $c = (i, j)$ *then let* $c' =$ *box of min* $\{P_{i+1,j}, P_{i,j+1}\}$
(ii) *Slide* $P_{c'}$ *into box* $c$ *and make* $c = c'$

*The resulting tableau is* $j^c(P)$*.*

*A* backward slide *from box* $c$ *on* $P$ *is performed as follows :*

(1) *Pick* $c$ *to be an outer corner of* $\lambda$
(2) *While* $c$ *is not an outer corner of* $\mu$ *, do*
(i) *If* $c = (i, j)$ *then let* $c' =$ *box of max* $\{P_{i-1,j}, P_{i,j-1}\}$
(ii) *Slide* $P_{c'}$ *into box* $c$ *and make* $c = c$*'*

*The resulting tableau is* $j_c(P)$*.*

Note : Sliding is an invertible operation. If $c$ is the box for forward slide on $P$ and box vacated by the slide is $d$, then a backward slide into $d$ restores $P$.
i.e $j_d j^c(P) = P$.
Also, $j^c j_d(P) = P$.

$$\text{Let P} = \begin{array}{ccccc} & & & 6 & 8 \\ & 2 & 4 & 5 & 9 \\ 1 & 3 & 7 & & \end{array}$$

Forward slide at c $= (1, 3)$.

$$\begin{array}{ccccc} & & \bullet & 5 & 8 \\ & 2 & 4 & 6 & 9 \\ 1 & 3 & 7 & & \end{array} \longrightarrow \begin{array}{ccccc} & & 4 & 5 & 8 \\ & 2 & \bullet & 6 & 9 \\ 1 & 3 & 7 & & \end{array} \longrightarrow \begin{array}{ccccc} & & 4 & 5 & 8 \\ & 2 & 6 & \bullet & 9 \\ 1 & 3 & 7 & & \end{array} \longrightarrow \begin{array}{ccccc} & & 4 & 5 & 8 \\ & 2 & 6 & 9 & \bullet \\ 1 & 3 & 7 & & \end{array}$$

This is $j^c(P)$
Backward slide at d $= (3, 5)$.

$$\begin{array}{ccccc} & & 4 & 5 & 8 \\ & 2 & 6 & 9 & \bullet \\ 1 & 3 & 7 & & \end{array} \longrightarrow \begin{array}{ccccc} & & 4 & 5 & 8 \\ & 2 & 6 & \bullet & 9 \\ 1 & 3 & 7 & & \end{array} \longrightarrow \begin{array}{ccccc} & & 4 & 5 & 8 \\ & 2 & \bullet & 6 & 9 \\ 1 & 3 & 7 & & \end{array} \longrightarrow \begin{array}{ccccc} & & \bullet & 5 & 8 \\ & 2 & 4 & 6 & 9 \\ 1 & 3 & 7 & & \end{array}$$

This is $j_d(P)$.
Clearly, $j_d j^c(P) = P$ . Also, it can be shown that $j^c j_d(P) = P$.

**Definition 4.2.** *A sequence of cells $(c_1, c_2, ...c_n)$ is a* slide sequence *for a tableau $P$ if we can legally form $P = P_0, P_1, P_2...P_l$ where $P_i$ is obtained from $P_{i-1}$ by performing a slide into cell $c_i$.*
*Partial tableaux $P$ and $Q$ are equivalent i.e $P \cong Q$ if $Q$ can be obtained from $P$ by some sequence of slides.*

**Lemma 4.3.** *Let $a_1 < a_2 < ...a_n$*

(1) *If $x < a_1$, then $a_1 a_2 ... a_n x \overset{K}{\cong} a_1 x a_2 ... a_n$*
(2) *If $x > a_n$, then $x a_1 a_2 ... a_n \overset{K}{\cong} a_1 a_2 ... a_{n-1} x a_n$*

**Proposition 4.4.** *If $P$ and $Q$ are standard skew tableaux,*

$$P \cong Q \Longrightarrow P \overset{K}{\cong} Q$$

.

*Proof.* By induction it suffices to prove the theorem when $P$ and $Q$ differ by one move( either a forward slide or a backward slide). If the move is horizontal, then clearly $\pi_P = \pi_Q$. If the move is vertical, then we can restrict to the case where P and Q have two rows. Let x be the element being moved and that $R_l$ and $S_l$ ( respectively $R_r$ and $S_r$ ) are left ( respectively right ) portions of the two rows. In $P$ say $x$ was in between $R_l$ and $R_r$ and sliding is done from box just below $x$. In $Q$, $x$ has slid down and a box between in $R_l$ and $R_r$ is empty.

Now induction is done on the number of elements in $P$ (or $Q$). If both contain only $x$, we are done. If not, suppose $|R_r| > |S_r|$. Let $y$ be the rightmost element of $R_r$ and let $P'$ and $Q'$ be $P$ and $Q$ respectively with $y$ removed. By our assumption, $P'$ and $Q'$ are still skew tableaux. So, applying induction yields

$$\pi_P = \pi_{P'y} \overset{K}{\cong} \pi_{Q'y} = \pi_Q$$

Similarly if $|S_l| > |R_l|$, we are done.

So we just need to consider the case when number of elements in $R_r$ is equal to number of elements in $S_r$ and number of elements in $R_l$ is same as number of elements in $S_l$. Say,

$$R_l = x_1 x_2 ... x_j$$

$$R_r = y_1 y_2 ... y_k$$

$$S_l = z_1 z_2 ... z_j$$

$$S_r = w_1 w_2 ... w_k$$

Either $j > 0$ or $k > 0$. Let us consider $j > 0$.

Since rows and columns of P increases, we have

$\pi_P = z_1 \dots z_j w_1 \dots w_k x_1 \dots x_j x y_1 \dots y_k$
$\Rightarrow \pi_P \cong z_1 x_1 z_2 \dots z_j w_1 \dots w_k x_2 \dots x_j x y_1 \dots y_k$ (By part 1 of Lemma 4.3)
$\Rightarrow \pi_P \cong z_1 x_1 z_2 \dots z_j x w_1 \dots w_k x_2 \dots x_j y_1 \dots y_k$ (By induction)
$\Rightarrow \pi_P \cong z_1 z_2 \dots z_j x w_1 \dots w_k x_1 x_2 \dots x_j y_1 \dots y_k$ (By part 1 of Lemma 4.3)
$\Rightarrow \pi_P \cong \pi_Q$

□

Illustration using an example : Let P = $\begin{array}{ccc} & & 4 \\ & 2 & 5 \\ 3 & 6 & \\ 7 & & \end{array}$

We perform sliding on this skew tableau and convert it into a normal tableau Q. The conversion is done as follows :-

$$\begin{array}{ccc} & \bullet & 4 \\ & 2 & 5 \\ 3 & 6 & \\ 7 & & \end{array} \longrightarrow \begin{array}{ccc} & 2 & 4 \\ & \bullet & 5 \\ 3 & 6 & \\ 7 & & \end{array} \longrightarrow \begin{array}{ccc} & 2 & 4 \\ \bullet & 5 & \\ 3 & 6 & \\ 7 & & \end{array} \begin{array}{ccc} & 2 & 4 \\ 3 & 5 & \\ \bullet & 6 & \\ 7 & & \end{array} \longrightarrow \begin{array}{ccc} & 2 & 4 \\ 3 & 5 & \\ 6 & \bullet & \\ 7 & & \end{array} \longrightarrow \begin{array}{ccc} 2 & \bullet & 4 \\ 3 & 5 & \\ 6 & & \\ 7 & & \end{array} \longrightarrow Q = \begin{array}{cc} 2 & 4 \\ 3 & 5 \\ 6 & \\ 7 & \end{array}$$

Thus, $P \cong Q$.
Also, $\pi_P = (736254)$ and $\pi_Q = (763524)$
Insertion tableau for $\pi_Q$ is $Q$ itself. Also, it turns out that insertion tableau for $\pi_P$ is $Q$. Thus $\pi_P \stackrel{P}{\cong} \pi_Q$. By Theorem 3.5, we get $\pi_P \stackrel{K}{\cong} \pi_Q$. i.e. $P \stackrel{K}{\cong} Q$.

**Definition 4.5.** *Given a partial skew tableau $P$, we play* Jeu de Taquin *by choosing an arbitrary slide sequence that brings $P$ to normal shape and then applying the slides. Resulting tableau is $j(P)$. In fact, $j(P)$ is independent of the slide sequence .*

**Theorem 4.6.** *If $P$ is a partial skew tableau that is brought to a normal tableau $P'$ by slides, then $P'$ is unique. In fact, $P'$ is the insertion tableau for $\pi_P$*

*Proof.* By Proposition 4.4, $\pi_P \stackrel{K}{\cong} \pi_{P'}$.
So,by Theorem 3.5, $\pi_P$ and $\pi_{P'}$ have the same insertion tableau. Finally, Lemma 3.4 tells us that insertion tableau for $\pi_{P'}$ is P' itself. □

Finally, we state the main theorem of this article.

**Theorem 4.7.** *[Theorem 3.7.8 [1]] Let $P$ and $Q$ be partial skew tableaux. Then*
$P \cong Q \Longleftrightarrow P \stackrel{K}{\cong} Q$.

*Proof.* The forward direction is already proved in Proposition 4.4. For the other implication, since $P \stackrel{K}{\cong} Q$, their row words must have the same $P$ tableaux ( From Theorem 3.5 ). So by the previous theorem, $j(P) = j(Q) = P'$(say). Thus we take $P$ to $P'$ and then the inverse of the sequence taking $Q$ to $P'$. Hence $P \cong Q$. □

## 5. Conclusion

Thus we have given a characterisation of Knuth equivalence in terms of slides.

## References

[1] Sagan,Bruce E., The symmetric group: representations, combinatorial algorithms, and symmetric functions, Graduate Texts in Mathematics, Vol. 203, Second edition, Springer-Verlag, New York, 2001.

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