22 January 2014

Representations of symmetric groups
Lecture 6

In the last lecture we calculated the character table of S_3. Let us analyze the irreducible representation corresponding to the last row.

Since S_3 is the group of permutations on 3 elements there is an action of S_3 on $V = \mathbb{C}^3$ which permutes coordinates. If $\{e_1, e_2, e_3\}$ is the standard basis, then $\rho_g(e_i) := e_{g(i)}$. Let $\rho : S_3 \to \mathrm{GL}(V)$ denote the corresponding representation.

It is easy to see the subspace $W = \mathrm{span}(e_1+e_2+e_3)$ is a S_3-invariant subspace of V. We know from general theory that there exists an S_3-invariant complement $W^\perp$ of W. This complement can be constructed by taking the kernel of the projection of V onto W. Then $W^\perp = \{(a, b, c) \mid a+b+c = 0\}$.

Thus, $V = W \oplus W^\perp$. The representation W is infact isomorphic to the trivial representation. For $w = (w_1, w_2, w_3) \in W$ we have that $w_1 = w_2 = w_3$ so that $\rho_g^W(w) = w$ for every $w \in W$. Thus $\rho_r^W = e$.

Let $f_1 = (1, 1, 1)$ be a basis of W and let us choose a basis for $W^\perp$ as, say $f_2 = (0, 1, -1), f_3 = (1, 0, -1)$. The $\{f_1, f_2, f_3\}$ forms a basis of V.

Claim: $W^\perp$ is an irreducible representation of G.

If we can show that for some $g \in G$, the g-invariant subspaces of V are not preserved by some other element $h \in G$, then we are done, since this would show that there is no way to decompose V into G-invariant subspaces.

Let $g = (1\ 3\ 2) \in S_3$. W.r.t. the basis $\{f_2, f_3\}$ of $W^\perp$, $\rho_g^{W^\perp} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$. The eigenvalues of $\rho_g^{W^\perp}$ are $\{\omega = \frac{-1+i\sqrt{3}}{2}, \overline{\omega} = \frac{-1+i\sqrt{3}}{2}\}$ and suppose the (linearly independent) eigenvectors are v_1 and v_2. The eigenspaces U_1 and U_2 spanned by v_1 and v_2 respectively are g-invariant subspaces of $W^\perp$.

Let $h = (1\ 2)$, $\rho_h^{W^\perp} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$\begin{aligned}
\rho_g^{W^\perp}(\rho_h^{W^\perp v_1)} &= \rho_{gh}^{W^\perp(v_1)} \\
&= \rho_{hg^{-1}}^{W^\perp}(v_1) \text{ (since } gh = hg^{-1}) \\
&= \rho_h^{W^\perp}(\rho_{g^{-1}}^{W^\perp} v_1) \\
&= \rho_h^{W^\perp}(\overline{\omega} v_1) \\
&= \overline{\omega}(\rho_h^{W^\perp} v_1).
\end{aligned}$$

Thus $\rho_h^{W^\perp} v_1 \in U_2$. Thus $\rho_h^{W^\perp}$ interchanges the subspaces U_1 and U_2. Therefore, there is no G-invariant subspace of $W^\perp$.

Thus, $W^\perp$ is a 2-dim'l irreducible representation of S_3. It is called the 'standard' or 'defining' representation of S_3.

The triangle representation: S_3 acts on the vertices of an equilateral triangle in $\mathbb{R}^2$ by permuting its vertices, i.e. as isometries of $\mathbb{R}^2$. Now $S_3 \subset \mathrm{GL}_2(\mathbb{R}) \subset \mathrm{GL}_2(C)$, thus we get a corresponding action of S_3 on $\mathbb{C}^2$. This is a 2-dim'l representation called the triangle representation of S_3 on $\mathbb{C}^2$. We will denote it by T.

The correspondence between elements of S_3 and isometries of the equilateral triangle are given as follows:

$$e \leftrightarrow \text{id},\ 2\text{-cycles} \leftrightarrow \text{reflections},\ 3\text{-cycles} \leftrightarrow \text{rotations}.$$

Claim: T is an irreducible representation of S_3.

Let A denote the matrix of rotation through $120°$ (this corresponds to the permutation $g = (1\,2\,3)$ in S_3). Then $A^3 = I$, which implies $A^2 + A + I = 0$ (since $A \neq I$). Let ω and $\overline{\omega}$ be the distinct eigenvalues of A and v_1, v_2 denote the corresponding eigenvectors. Now let B denote the reflection matrix corresponding to the permutation $(1\ 2) \in S_3$. Check that B interchanges the eigenspaces of A (i.e. $Bv_1 \in \text{span}(v_2)$ and $Bv_2 \in \text{span}(v_1)$). Thus there are no S_3-invariant subspaces of T and we are done.

By uniqueness, T is isomorphic to W_2. For another proof, consider the action of S_3 on the triangle in W_2 having vertices $(2,-1,-1)$, $(-1,2,-1)$, $(-1,-1,2)$.

TABLE 1. g and corresponding ρ_g for each irreducible representation of S_3

g	Trivial repr.	Sign repr.	Standard repr. W_2	Standard repr. T
e	1	1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$(1\ 2)$	1	-1	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$(1\ 3)$	1	-1	$\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$
$(2\ 3)$	1	-1	$\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$
$(1\ 2\ 3)$	1	1	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$
$(1\ 3\ 2)$	1	1	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$

6.1. **The standard representation of S_n.** The following lemma is easy to prove:

Lemma 6.1. *For a representation V of a finite group G, let $V = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$ be the decomposition of V into distinct irreducibles. Then*

$$\dim\text{Hom}_G(V,V) = \sum a_i^2.$$

Proof. Note that if U is an irreducible representation of G then

- $\text{Hom}_G(U^{\oplus p}, U) = \text{Hom}_G(U,U)^{\oplus p} \cong \mathbb{C}^p$.
- $\text{Hom}_G(U^{\oplus p}, U^{\oplus q}) \cong \mathbb{C}^{pq}$

□The symmetric group $G = S_n$ acts on $V = \mathbb{C}^n$ by permuting coordinates giving rise to the permutation representation. Let $W_1 = \text{span}\{(1,1,\ldots,1)\}$, then W_1 is a 1-dim'l G-invariant subspace of V. Let

$$W_2 = W_1^{\perp} = \{(v_1,\ldots,v_n) \in V \mid v_1 + \cdots + v_n = 0\}.$$

Then W_2 is also a G-invariant subspace of V of dimension $n-1$.

Claim: W_2 is irreducible.

Let $A = (a_{ij})$ be the matrix form of an element of $\text{Hom}_G(V,V)$ w.r.t some basis of V. Then A is G-linear which means A commutes with each $\rho_g \in \text{Hom}_G(V,V)$. Since V is the permutation representation, each ρ_g is a permutation matrix. Thus A permutes with all $(n-1)\times(n-1)$ permutation matrices. As a result, $a_{ii} = a_{jj}$ for all $i \neq j$ and $a_{ij} = a_{kl}$ for all $i \neq k, j \neq l$. This gives

$$A = \begin{pmatrix} a & b & \ldots & b \\ b & a & \ldots & b \\ b & b & \ddots & b \\ b & b & \ldots & a \end{pmatrix} = aI + b\begin{pmatrix} 0 & 1 & \ldots & 1 \\ 1 & 0 & \ldots & 1 \\ 1 & 1 & \ddots & 1 \\ 1 & 1 & \ldots & 0 \end{pmatrix}.$$

Thus we have found a basis for $\mathrm{Hom}_G(V, V)$ implying that $\dim\mathrm{Hom}_G(V, V) = 2$. It follows from Lemma 6.1 that V can have only two irreducible summands, each occuring with multiplicity 1. Thus W_2 is irreducible.

Thus for each S_n we have a $(n-1)$-dimensional irreducible representation, called the *standard* or *defining* representation of S_n. Just as in the case of S_3, the standard representation of S_n can be realized geometrically as the action of S_n on the $(n-1)$-simplex in $\mathbb{R}^n$ (by permuting vertices).

We can write a nice formula for the character of the standard representation. $V = W_1 \oplus W_2$ implies $\chi_V = \chi_{W_1} + \chi_{W_2}$ so that

$$\begin{aligned} \chi_{W_2}(g) &= \chi_V(g) - \chi_{W_1}(g) \\ &= \#\text{ of fixed points of } g - 1. \end{aligned}$$

Example 6.2. *Symmetric group S_4: The first three rows in the character table are easy to write.*

	e	$(1\ 2)$	$(1\ 2\ 3)$	$(1\ 2\ 3\ 4)$	$(1\ 2)(3\ 4)$
V_0	1	1	1	1	1
V_1	1	-1	1	1	1
V_2	3	1	0	-1	-1

In the next lecture we will calculate and analyze the 2 *remaining rows of this table.*