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Representations of symmetric groups
Lecture 5

Today we want to show that the number of irreducible representations of G is equal to the number of conjugacy classes of G.

For this, we need the following theorem which says that a linear combination $\sum \alpha(g)\rho_g$ of elements of $\mathrm{GL}(V)$ is G-linear for all V iff α is a class function on G. Using this result we will be able to show that the set of irreducible characters of G forms a basis of $C(G)$.

Theorem 5.1. *Let $\alpha : G \to \mathbb{C}$ be any function on G and for any representation V of G let $\phi_{\alpha,V} = \sum \alpha(g)\rho_g$ be an element of $\mathrm{GL}(V)$. Then $\phi_{\alpha,V}$ is G-linear for all V if and only if α is a class function.*

Proof. Suppose α is a class function. To prove that $\phi_{\alpha,V}$ is G-linear, we must show that $\phi_{\alpha,V} \circ \rho_h = \rho_h \circ \phi_{\alpha,V}$ for all $h \in G$.

Consider

$$\begin{aligned}
(\phi_{\alpha,V} \circ \rho_h)(v) &= \sum \alpha(g)\rho_g\rho_h(v) \\
&= \sum \alpha(hgh^{-1})\rho_h\rho_g\rho_{h^{-1}}\rho_h(v) \\
&= \sum \alpha(hgh^{-1})\rho_h\rho_g(v) \\
&= \rho_h(\sum \alpha(hgh^{-1})\rho_g(v)) \\
&= (\rho_h \circ \phi_{\alpha,V})(v)
\end{aligned}$$

Thus $\phi_{\alpha,V}$ is G-linear for all V.

Conversely, suppose $\phi_{\alpha,V}$ is G-linear for all V and suppose α is not a class function. Then

$$(\phi_{\alpha,V} \circ \rho_h)(v) = \sum \alpha(g)\rho_{gh}(v)$$

while

$$\begin{aligned}
(\rho_h \circ \phi_{\alpha,V})(v) &= \rho_h(\sum \alpha(g)\rho_g(v)) \\
&= \sum \alpha(g)\rho_{hg}(v)
\end{aligned}$$

so that $\phi_{\alpha,V} \circ \rho_h \neq \rho_h \circ \phi_{\alpha,V}$, a contradiction.

(In particular, let V be the regular representation of G and $v = e_{h^{-1}}$, then LHS$= \sum \alpha(g)\rho_{gh}(e_{h^{-1}}) = \sum \alpha(g)e_g$ while RHS$= (\rho_h \circ \phi_{\alpha,V})(v) = \sum \alpha(g)e_{hgh^{-1}}$). ☐

Lemma 5.2. *A representation V of G is irreducible if and only if V^* is irreducible.*

Proof.

$$\begin{aligned}
V \text{ irreducible} &\iff \langle \chi_V, \chi_V \rangle = 1 \\
&\iff \overline{\langle \chi_V, \chi_V \rangle} = 1 \\
&\iff \langle \overline{\chi_V}, \overline{\chi_V} \rangle = 1 \\
&\iff \langle \chi_{V^*}, \chi_{V^*} \rangle = 1 \\
&\iff V^* \text{ irreducible.}
\end{aligned}$$

☐

Corollary 5.3. *The set of irreducible representations forms an orthonormal basis of $C(G)$.*

Proof. Let $\alpha \in C(G)$ and let $\langle \alpha, \chi_V \rangle = 0$ for every irreducible representation V of G.

Claim: $\alpha = 0$.

Consider $\phi_{\alpha,V} = \sum \alpha(g)\rho_g : V \to V$. By Schur's lemma, V irreducible implies that $\phi_{\alpha,V} = \lambda I$ for some $\lambda \in \mathbb{C}$. If $\dim V = n$, $\mathrm{Tr}(\phi_{\alpha,V}) = n\lambda$. Thus

$$\begin{aligned}
\lambda &= \frac{1}{n}\mathrm{Tr}(\phi_{\alpha,V}) \\
&= \frac{1}{n}\sum \alpha(g)\mathrm{Tr}(\rho_g) \\
&= \frac{1}{n}\sum \alpha(g)\chi_V(g) \\
&= \frac{1}{n}\overline{\sum \overline{\alpha(g)}\chi_{V^*}(g)} \\
&= \frac{|G|}{n}\overline{\langle \alpha, \chi_{V^*}\rangle} \\
&= 0 \text{ (since } \langle \alpha, \chi_V\rangle = 0 \text{ for all irreducible } V).
\end{aligned}$$

Thus, $\lambda = 0$, so $\phi_{\alpha,V} = 0$ which implies that $\sum \alpha(g)\rho_g = 0$ for any representation V of G. In particular, this is true for the regular representation R of G. But in R, the elements $\{e_g\}_{g\in G}$ are linearly independent, i.e. the elements ρ_g are linearly independent as elements of $\mathrm{End}(R)$. Thus $\alpha(g) = 0$ for all $g \in G$ implying $\alpha = 0$. □

Note that $C(G)$ has a basis of functions which are 1 on a given conjugacy class and 0 on the others. Hence $\dim C(G)$ equals the number of conjugacy classes of G. Putting everything together we have:

Corollary 5.4. *The number of irreducible representations of G is equal to the number of conjugacy classes of G.*

Proof. # irreducible representations of $G = \dim C(G) =$ # conjugacy classes of G. □

5.1. **The regular representation of G.** Recall that for any finite group G, we can define the regular representation as the vector space R spanned by basis $\{e_s\}_{s\in G}$. The action of G on R is defined by $\rho_t(e_s) = e_{ts}$. Thus R is a special case of the permutation representation.

When is R irreducible? Problem 2 on HW 2 tells us that $\chi_R(g) =$ # fixed points of g, so that

$$\chi_R(g) = \begin{cases} |G| & g = e \\ 0 & g \neq e. \end{cases}$$

Thus R is irreducible iff $G = \{e\}$.

Suppose $R = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$, then

$$a_i = \langle \chi_{V_i}, \chi_R\rangle = \frac{1}{|G|}\sum \overline{\chi_{V_i}(g)}\chi_R(g) = \frac{1}{|G|}\chi_{V_i}(e)|G| = \dim V_i.$$

This gives us the following consequences:

(1) Every irreducible representation of G appears as a summand of the regular representation of G. (In particular, this tells us again that there are only finitely many irreducible representations).
(2) Every irreducible representation appears in R with multiplicity equal to its dimension.
(3) We know that $|G|$ equals the degree of the regular representation. Thus we get

$$|G| = \dim R = \sum a_i \dim V_i = \sum \dim V_i^2. \tag{5.1}$$

(4) If $g \neq e$, then

$$\sum (\dim V_i)\chi_{V_i}(g) = \sum a_i \chi_{V_i}(g) = \chi_R(g) = 0. \tag{5.2}$$

Equations 5.1 and 5.2 are useful in calculating an unknown character if all but one characters is known.

5.2. **List of properties of characters.**

(1) χ_V is constant on the conjugacy classes of G.
(2) $\chi_V(1) = \dim V$.
(3) The irreducible characters of G form an orthonormal basis of $C(G)$ w.r.t the inner product

$$\langle \chi_V, \chi_W \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)} \chi_W(g).$$

(4) If $V = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$ is the decomposition into distinct irresucibles V_i, then $a_i = \langle \chi_V, \chi_{V_i} \rangle$.
(5) If R is the regular representation of G, then

$$\chi_R(g) = \begin{cases} |G| & g = e \\ 0 & g \neq e. \end{cases}$$

(6) Every irreducible representation V_i shows up as a summand of R with multiplicity dim V_i.
(7) $|G| = \sum(\dim V_i)^2 = \sum(\chi_{V_i}(e))^2$, where V_i are distinct irreducible representations of G. In particular, this implies that the sum of squares of elements in the first column of the character table add up to $|G|$. This is useful in finding the possible dimensions of irreducible representations of G.
(8) # irreducible representations of G are equal to # conjugacy classes of G. Thus, in the character table of G, the number of rows equals the number of columns.

5.3. **Examples.**

Example 5.5. *Cyclic group C_n: Suppose $C_n = \{g, g^2, \dots, g_{n-1}, g^n = e\}$. As in any abelian group, each element of C_n is a conjugacy class in itself. Thus C_n has n conjugacy classes. We have seen that any irreducible representation of C_n is 1-dim'l and corresponds to a nth root of unity. Let V_k be the irreducible representation corresponding to ω^k, where $\omega = e^{\frac{2\pi i}{n}}$. Then $V_0, V_1, \dots V_{n-1}$ are all the irreducible representations of C_n. The character table of C_n is as follows: In particular, let us*

	e	g	$\dots$	g^k	$\dots$	g^{n-1}
V_0	1	1	$\dots$	1	$\dots$	*1*
V_1	1	ω	$\dots$	ω^k	$\dots$	ω^{n-1}
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
V_l	1	ω^l	$\dots$	$\omega^l k$	$\dots$	$\omega^{l(n-1)}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
V_{n-1}	1	ω^{n-1}	$\dots$	$\omega^{(n-1)k}$	$\dots$	$\omega^{(n-1)^2}$

look at C_4. The four fourth roots of unity are:

$$\omega = e^{\frac{2\pi i}{4}} = \cos(\frac{2\pi}{4}) + i\sin(\frac{2\pi}{4}) = i,\ \omega^2 = -1,\ \omega^3 = -i,\ \omega^4 = 1.$$

	e	g	g^2	g^3
V_0	*1*	*1*	*1*	*1*
V_1	*1*	*i*	*-1*	*-i*
V_2	*1*	*-1*	*1*	*-1*
V_3	*1*	*-i*	*-1*	*i*

Next, we look at the smallest non-abelian group.

Example 5.6. *Symmetric group S_3: What are the 1-dim'l representations of S_3? There are 2 candidates:*

- *the trivial representation $\rho^1_g = 1$ for every $g \in S_3$;*
- *the 'sign representation' $\rho^2_g = sgn(g)$ for every $g \in S_3$.*

Let us write elements of S_3 *in cycle notation as:*

$$S_3 = \{e = (1)(2)(3), (1\ 2), (1\ 3), (2\ 3), (1\ 3\ 2), (1\ 2\ 3)\}.$$

There are three conjugacy classes: $\{e\}, \{(1\ 2), (1\ 3), (2\ 3)\}, \{(1\ 3\ 2), (1\ 2\ 3)\}$ *(cycles of the same length are conjugate to each other).*

With this information, it is already possible to calculate the character table of S_3*. We know the first two rows:*

	e	$(1\ 2)$	$(1\ 2\ 3)$
V_0	*1*	*1*	*1*
V_1	*1*	*-1*	*1*
V_2	a	b	c

The third row can be calculated using the properties of characters. We know that

$$|S_3| = 6 = \textit{sum of squares of elements of first column} = 1 + 1 + a^2.$$

Thus, $a = \dim V_2 = 2$.

Further, $\sum(\dim V_i)\chi_{V_i}(g) = 0$ *implies:*

- *for the second column:* $\dim V_0(1) + \dim V_1(-1) + \dim V_2(b) = 0 \implies 1 - 1 + 2b = 0 \implies b = 0$.
- *for the third column:* $\dim V_0(1) + \dim V_1(1) + \dim V_2(c) = 0 \implies 1+1+2c = 0 \implies c = -1$.

Thus the character table of S_3 *is:*

	e	$(1\ 2)$	$(1\ 2\ 3)$
V_0	1	1	1
V_1	1	-1	1
V_2	2	0	-1