Viennot Geometric Construction

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Abstract : Our main goal in this article is to prove the theorem of Schützenberger. In first two sections we establish Robinson Schensted correspondence using Schensted algorithm. We later appeal to Viennot's Geometric Construction and establish this correspondence again. And we finally prove the theorem of Schützenberger.

§1 Basic Definitions

Definition 1.1 A *partition* of a positive integer n is a sequence $\lambda = (\lambda_1, .., \lambda_k)$ where λ_i weakly decreasing sequence of positive integers and $\sum \lambda_i = n$. It is denoted by $\lambda \vdash n$.
Example: (2,1,1) is a partition of 4.
Notation: λ denotes a partition unless otherwise specified.

Definition 1.2 Suppose $\lambda = (\lambda_1, .., \lambda_k) \vdash n$. Then a Young diagram of shape λ is an array of k left-justified rows with λ_i dots in i^{th} row.

As an example the partition (4,3,1) has Young diagram
$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & & & \end{array}$$

Definition 1.3 Let $\lambda \vdash n$. A *generalized Young tableau* of shape λ is an array obtained by replacing the dots of young diagram of shape λ by positive integers.

An example of generalized Young tableau of shape (5,3,2) is
$$\begin{array}{ccccc} 1 & 2 & 2 & 3 & 9 \\ 4 & 2 & 5 & & \\ 3 & 5 & & & \end{array}$$

Notation: For every Young tableau P , sh(P) denotes shape of P.

Definition 1.4 Suppose $\lambda \vdash n$. A *tableau* of shape λ is a generalized Young tableau in which all entries are distinct.

Definition 1.5 Let $\lambda \vdash n$. A *Standard tableau* of shape λ is a tableau of shape λ such that
1. Entries are numbers between 1 and n .
2. Rows and columns are increasing sequences.

Example:
$$\begin{array}{ccc} 1 & 2 & 4 \\ 3 & 7 & 8 \\ 5 & 9 & \\ 6 & & \end{array}$$
is a standard tableau of shape (3,3,2,1)

Definition 1.6 A tableau is called *partial tableau*, if the entries strictly increase along each row and strictly increase down each column .

Example: $\begin{array}{cccc} 1 & 3 & 6 & 8 \\ 4 & 5 & & \\ 7 & & & \\ 9 & & & \end{array}$ is a partial tableau of shape (4,2,1,1)

Definition 1.7 A *partial permutation* on a finite set S is a bijection between two specified subsets of S.

Example: $\pi = \begin{pmatrix} 1 & 2 & 4 & 7 \\ 2 & 5 & 7 & 8 \end{pmatrix}$

§2 The Robinson-Schensted correspondence

The Robinson Schensted correspondence is a bijective correspondence between elements of S_n and pair of standard Young tableaux of the same shape λ as λ varies over all partitions of n. The correspondence had been described by Robinson in 1938, in an attempt to prove the Little wood-Richardson rule. Later Schensted established the correspondence independently in 1961 using Schensted algorithm. Later R-S correspondence was generalised to R-S-K correspondence by Knuth. The simplest description of the R-S correspondence was given by the Schensted and we present it in this article. The correspondence is usually denoted as $\pi \overset{\text{R−S}}{\longleftrightarrow} (P(\pi),Q(\pi))$. This correspondence gives a combinatorial proof of the enumerative identity $\sum_{\lambda \vdash n} (t_\lambda)^2 = n!$, where t_λ is the number of tableaux of shape λ.

To establish R-S correspondence we first construct a map from S_n to pair of standard young tableaux of same shape which we denote by $\pi \xrightarrow{R-S} (P(\pi), Q(\pi))$. Then we prove $\pi \xrightarrow{R-S} (P(\pi), Q(\pi))$ gives the required correspondence. We construct the map, using $\pi \xrightarrow{R-S} (P(\pi), Q(\pi))$ the algorithms
i) row insertion and ii) placement of an element.

Algorithm 1: Row Insertion

The row insertion or row bumping is a procedure that takes a partial tableau P and a positive integer x different from the entries of P, and constructs a tableau denoted as $P \leftarrow x$ (or) $r_x(P)$. The algorithm is described as follows.

Let P be a partial tableau and x be a positive integer which is different from the entries of P. Then do the following

RS. Set $R :=$ the first row of P.

RS2 While x is less than some element of row R, do

RSa Let y be the smallest element of R greater than x and replace y by x in R .

RSb Set $x := y$ and $R :=$ the next row down.

RS3 Now x is greater than every element of R, so place x at the end of row R and stop.

Illustration:

Consider the partial tableau $P = \begin{array}{cccc} 1 & 2 & 5 & 8 \\ 3 & 7 & & \\ 6 & & & \\ 9 & & & \end{array}$

We insert 4 in P using the Row insertion algorithm

$$\begin{array}{llll} 1 & 2 & 5 & 8 \\ 3 & 7 & & \\ 6 & & & \\ 9 & & & \end{array} \leftarrow 4 \qquad \begin{array}{llll} 1 & 2 & \mathbf{4} & 8 \\ 3 & 7 & & \\ 6 & & & \\ 9 & & & \end{array} \begin{array}{l} \\ \leftarrow 5 \\ \\ \\ \end{array} \qquad \begin{array}{llll} 1 & 2 & 3 & 8 \\ 3 & \mathbf{5} & & \\ 6 & & & \\ 9 & & & \end{array} \begin{array}{l} \\ \\ \leftarrow 7 \\ \\ \end{array} \qquad \begin{array}{llll} 1 & 2 & 3 & 8 \\ 3 & 5 & & \\ 6 & \mathbf{7} & & \\ 9 & & & \end{array}$$

The fact that $P \leftarrow x$ is a partial tableau, if the same holds for P is not obvious from the procedure described above. Here I will outline the arguments that need to verified and supply proof for few statements.

Lemma 2.1 Let P be a partial tableau and $x \notin P$ be a positive integer. Then $r_x(P)$ is a tableau.
Proof: If no two rows of P have same length then there is nothing to prove. Consider the case where two rows of P have same length. Suppose a number is displaced from the first row, then are two possibilities, it can displace the number below it or to the left of it. It can't go to the right because the rows and columns are strictly increasing.

Lemma 2.2 Let P be a partial tableau and $x \notin P$ be a positive integer. Suppose that during row insertion the elements x_1, x_2, x_3,... are bumped from cells $(1, j_1), (2, j_2), (3, j_3)$,.... respectively. Let $P' = P \leftarrow x$. Then
1. $x < x_1 < x_2 <$ ....
2. $j_1 \geq j_2 \geq j_3 \geq$ ....
3. $P'_{i,j} \leq P_{i,j}$ for all i, j .

Proof: First and the last assertions are immediate from the description of row insertion. Suppose x_k is displaced from the cell (k, j_k). Since P is a partial tableau we have $P_{k+1,j} > P_{k,j} > x_k$ for all $j \geq j_k$. So x_k cannot bump an element that is right to it. So it has to bump an element below it or to the left of it.

Theorem 2.3 Let P be a partial tableau and x be a positive integer distinct from the entries of P. Then $P \leftarrow x$ is also a partial tableau.
Proof: From lemma 2.1 we know that $P' = P \leftarrow x$ is a tableau. We need to show
(i) rows of $P \leftarrow x$ are strictly increasing and
(ii) columns of $P \leftarrow x$ are strictly increasing

Suppose that during row insertion the elements x_1, x_2, x_3, ... are bumped from cells $(1, j_1), (2, j_2), (3, j_3)$,.... respectively. When a number x_i is inserted into $(i+1)^{th}$ row the number left to it are less than x_i and the numbers right to it are greater than x_i , thus we have rows are strictly increasing. Further, it is clear that $P'_{i,n} = P_{i,n} < P_{i+1,n} = P'_{i+1,n}$ for all $n \neq j_i$, j_{i+1}. From lemma 2.2 we know that $j_i \geq j_{i+1}$.
If $j_i = j_{i+1}$, then $P'_{i,j_i} = x_{i-1} < x_i = P'_{i+1,j_{i+1}} = P'_{i+1,j_i}$.
If $j_i > j_{i+1}$, then $P'_{i,j_i} = x_{i-1} < x_i = P'_{i+1,j_{i+1}} < P'_{i+1,j_i}$. Further $P'_{i,j_{i+1}} < P'_{i,j_i} = P'_{i,j_{i+1}}$.
In either of the cases we have columns of P' are strictly increasing.

Algorithm 2: Placement of an element

Placement of an element in a tableau is even easier than insertion. Suppose that Q is a partial tableau of shape λ and that (i,j) is an outer corner of λ. Let k be greater than every element of Q. Then to place k in Q at cell (i,j), is merely set $Q_{ij} = k$. The restriction on k guarantees that the new array is still a take partial tableau. For example, consider

$$Q = \begin{array}{lll} 1 & 2 & 4 \\ 3 & 7 & \\ 5 & & \\ 9 & & \end{array} \text{, then placing } k = 8 \text{ in cell } (i,j) = (2,3) \text{ of } Q \text{ yields } \begin{array}{lll} 1 & 2 & 4 \\ 3 & 7 & 8 \\ 5 & & \\ 9 & & \end{array}$$

Finally, we are now in position to describe the Schensted algorithm which gives the map $\pi \xrightarrow{R-S} (P(\pi), Q(\pi)$.

Schensted Algorithm

Suppose that π is given in two-line notation as $\pi = \begin{pmatrix} 1 & 2 & ..... & n \\ x_1 & x_2 & ..... & x_n \end{pmatrix}$.

Given $\pi \in S_n$, Schensted algorithm constructs a sequence of tableaux pairs (P_0, Q_0), (P_1, Q_1),...... (P_n,Q_n) = (P,Q) such that sh(P_k) = sh(Q_k) for all 1≤ $k \leq n$. Set $(P_0, Q_0) = (\emptyset, \emptyset)$ i.e, P_0, Q_0 are empty tableaux. Assuming (P_{k-1}, Q_{k-1}) has been constructed, define (P_k, Q_k) by

$$\begin{aligned} P_k &= r_{x_k}(P_{k-1}) \\ Q_k &= \text{place } k \text{ into } Q_{k-1} \text{ at the cell } (i,\, j) \text{ where the insertion terminates.} \end{aligned}$$

It is clear that form theorem 2.3 that each k, (P_k, Q_k) are standard young tableaux and have same shape. Now we consider an example for illustrating the complete algorithm. Let

$$\pi = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \mathbf{4} & \mathbf{1} & \mathbf{3} & \mathbf{5} & \mathbf{7} & \mathbf{2} & \mathbf{6} \end{matrix}$$

Then the tableaux constructed by the algorithm are

$$\mathbf{P_k}: \quad \emptyset \;,\; \mathbf{4} \;,\; \begin{matrix} \mathbf{1}, \\ \mathbf{4} \end{matrix} \; \begin{matrix} \mathbf{1} & \mathbf{3} \\ \mathbf{4} & \end{matrix} \;,\; \begin{matrix} \mathbf{1} & \mathbf{3} & \mathbf{5} \\ \mathbf{4} & & \end{matrix} \;,\; \begin{matrix} \mathbf{1} & \mathbf{3} & \mathbf{5} & \mathbf{7} \\ \mathbf{4} & & & \end{matrix} \;,\; \begin{matrix} \mathbf{1} & \mathbf{2} & \mathbf{5} & \mathbf{7} \\ \mathbf{3} & & & \\ \mathbf{4} & & & \end{matrix} \;,\; \begin{matrix} \mathbf{1} & \mathbf{2} & \mathbf{5} & \mathbf{6} \\ \mathbf{3} & \mathbf{7} & & \\ \mathbf{4} & & & \end{matrix} \quad = \mathbf{P}$$

$$Q_k: \quad \emptyset \;,\; 1 \;,\; \begin{matrix} 1 \\ 2 \end{matrix} \;,\; \begin{matrix} 1 & 3 \\ 2 & \end{matrix} \;,\; \begin{matrix} 1 & 3 & 4 \\ 2 & & \end{matrix} \;,\; \begin{matrix} 1 & 3 & 4 & 5 \\ 2 & & & \end{matrix} \;,\; \begin{matrix} 1 & 3 & 4 & 5 \\ 2 & & & \\ 6 & & & \end{matrix} \;,\; \begin{matrix} 1 & 3 & 4 & 5 \\ 2 & 7 & & \\ 6 & & & \end{matrix} \quad = Q$$

So ,

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \mathbf{4} & \mathbf{1} & \mathbf{3} & \mathbf{5} & \mathbf{7} & \mathbf{2} & \mathbf{6} \end{matrix} \xrightarrow{R-S} \left(\begin{matrix} \mathbf{1} & \mathbf{2} & \mathbf{5} & \mathbf{6} \\ \mathbf{3} & \mathbf{7} & & \\ \mathbf{4} & & & \end{matrix} \;,\; \begin{matrix} 1 & 3 & 4 & 5 \\ 2 & 7 & & \\ 6 & & & \end{matrix} \right)$$

Therefore Schensted algorithm gives the required map $\pi \xrightarrow{R-S} (P(\pi), Q(\pi))$.

Inversion of Schensted Algorithm

The Schensted algorithm given in previous section can be inverted *i.e*, $\pi \xrightarrow{R-S} (P(\pi), Q(\pi))$ is invertible. This will enable us to prove R-S correspondence is a bijection as outlined earlier. We prove it in this section using the inversion algorithm.

Now we start with a pair of standard young tableaux (P, Q) of same shape and want to construct a permutation π such that $\pi \xrightarrow{R-S} (P, Q)$. The idea is simple, we merely reverse the preceding algorithm step by step. We begin by defining $(P_n, Q_n) = (P, Q)$. Assuming that (P_k, Q_k) has been constructed, we will find x_k (the k^{th} element of π) and (P_{k-1}, Q_{k-1}). To avoid double subscripting in what follows, we denote the (i,j) entry of P_k by P_{ij}. Find the cell (i, j) containing k in Q_k. Since this is the largest element in Q_k P_{ij} must have been the last element to be displaced in the construction of P_k. For convenience, we assume the existence of an empty zeroth row above the first row of P_k.

The following gives a systematic procedure to delete P_{ij} from P. This procedure is usually called as ***inversion algorithm***.

SR1 Set $x := P_{ij}$ and erase P_{ij}.

Set R := the $(i-1)^{th}$ row of P_k .

SR2 While R is not the zeroth row of P_k , do

SRa Let y be the largest element of R smaller than x and replace y by x in R

SRb Set $x := y$ and R := the next row up.

SR3 Now x has been removed from the first row, so set $x_k := x$.

It is easy to see that P_k becomes P_{k-1} after the deletion process just described is complete and Q_{k-1} is Q_k with the k erased.The other way of describing above is by row-inserting x_k in the tableau obtained we get P_k. Continuing in this way, we eventually recover all the elements of permutation π in reverse order $i.e$, $\pi = x_1\ x_2.....x_n$, where x_k are as described in inversion algorithm.

As an example apply inversion algorithm to the pair of tableaux we obtained in illustration of Schensted algorithm.

Example:

$$P_k: \begin{matrix}\mathbf{1} & \mathbf{2} & \mathbf{5} & \mathbf{6}\\ \mathbf{3} & \mathbf{7} & & \\ \mathbf{4} & & & \end{matrix}\ ,\ \begin{matrix}\mathbf{1} & \mathbf{2} & \mathbf{5} & \mathbf{7}\\ \mathbf{3} & & & \\ \mathbf{4} & & & \end{matrix}\ ,\ \begin{matrix}\mathbf{1} & \mathbf{3} & \mathbf{5} & \mathbf{7}\\ \mathbf{4} & & & \end{matrix}\ ,\ \begin{matrix}\mathbf{1} & \mathbf{3} & \mathbf{5}\\ \mathbf{4} & & \end{matrix}\ ,\ \begin{matrix}\mathbf{1} & \mathbf{3}\\ \mathbf{4} & \end{matrix}\ ,\ \begin{matrix}\mathbf{1}\\ \mathbf{4}\end{matrix}\ ,\ \mathbf{4}\ ,\ \emptyset$$

$$Q_k: \begin{matrix}1 & 3 & 4 & 5\\ 2 & 7 & & \\ 6 & & & \end{matrix}\ ,\ \begin{matrix}1 & 3 & 4 & 5\\ 2 & & & \\ 6 & & & \end{matrix}\ ,\ \begin{matrix}1 & 3 & 4 & 5\\ 2 & & & \end{matrix}\ ,\ \begin{matrix}1 & 3 & 4\\ 2 & & \end{matrix}\ ,\ \begin{matrix}1 & 3\\ 2 & \end{matrix}\ ,\ \begin{matrix}1\\ 2\end{matrix}\ ,\ 1\ ,\ \emptyset$$

Writing the erased entries in the reverse order we have $\pi = \ 4\ \ 1\ \ 3\ \ 5\ \ 7\ \ 2\ \ 6$

Thus from above discussion we have

Theorem 2.4([Rob 38, Sch 61]) The map $\pi \xrightarrow{R-S} (P(\pi),Q(\pi))$ is a bijection between elements of S_n and pairs of standard tableaux of the same shape λ, as λ varies over all partitions of n.

Definition 2.5 Two permutations π, $\sigma \in S_n$ are said to be P- equivalent, if $P(\pi) = P(\sigma)$. It is denoted as $\pi \overset{\text{P}}{\cong} \sigma$.

Example: $2\ 1\ 3\ 4 \overset{\text{P}}{\cong} 2\ 3\ 1\ 4$.

Definition 2.6 Let $x < y < z$. Two permutations π, $\sigma \in S_n$ are said to differ by a Knuth relation of the first kind, written $\pi \overset{1}{\cong} \sigma$ if for some k,

1. $\pi = x_1.....\ yxz\ .....\ x_n$ and $\sigma = x_1.....\ yzx\ .....\ x_n$ or vice versa.

They differ by a Knuth relation of the second kind, written $\pi \overset{2}{\cong} \sigma$, if for some k,

2. $\pi = x_1.....\ xzy\ .....\ x_n$ and $\sigma = x_1.....\ zxy\ .....\ x_n$ or vice versa.

The two permutations are Knuth equivalent, written as $\pi \overset{K}{\cong} \sigma$ if there is a sequence of permutations such that $\pi = \pi_1 \overset{i}{\cong} \pi_2 \overset{j}{\cong} ..... \overset{l}{\cong} \pi_k = \sigma$ where i, j,..., $l \in \{1, 2\}$.

Following theorem due to Knuth describes that Knuth equivalence and P-equivalence are same notions.

Theorem 2.7[Knu 70] If π, $\sigma \in S_n$ then $\pi \overset{\text{K}}{\cong} \sigma \iff \pi \overset{\text{P}}{\cong} \sigma$.

Proof: We skip proof of theorem as it is long. Reader may refer to theorem 3.4.3 of [Sagan].

§3 Viennot Geometric Construction

Viennot in the paper *Une forme géométrique de la correspondence de Robinson-Schensted (1977)* gave a beautiful geometric description of the Robinson-Schensted correspondence. This was later generalised to *matrix-ball construction* by Fulton. The matrix-ball construction can be used to establish R-S-K correspondence. We would discuss only the Viennot construction in this article and towards the end prove a remarkable theorem of Schützenberger which states taking the inverse of the permutation merely interchanges the two tableaux in R-S correspondence *i.e*, if

$$\pi \xrightarrow{R-S} (P(\pi),\, Q(\pi)) \text{ then } \pi^{-1} \xrightarrow{R-S} (Q(\pi),\, P(\pi)).$$

Consider a point $(i,\, j)$ in first quadrant of the Cartesian plane. Imagine a light shining from the origin so that each box casts shadow with boundaries parallel to the coordinate axes. For example, the shadow cast by the box at (3,4) is :

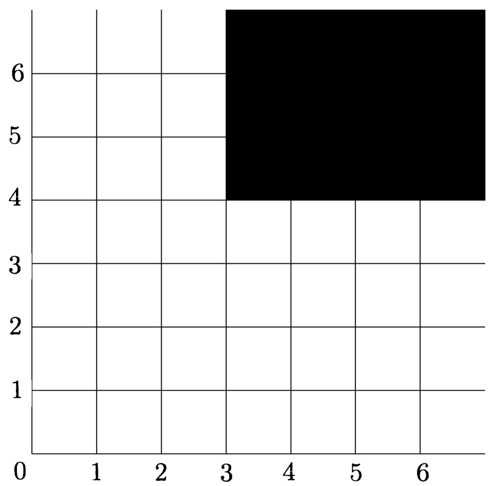

Figure 1: Shadow Diagram of point (3,4)

§§ 3.1 Shadow Lines and Shadow Diagram

Lemma 3.1.1 Let (i, j) be a point in first quadrant. Then (m, n) lies in the shadow of (i, j) if and only if $m \geq i$ and $n \geq j$.

Proof: It is an easy observation.

Above lemma characterizes the points which lie in shadow of point (i,j).

Let (i, j) and (m, n) be two points in first quadrant of the Cartesian plane. Define $(i, j) \trianglelefteq (m, n)$ if and only if (i, j) lies in the shadow of (m, n). Then it follows from lemma 3.1 that $\trianglelefteq$ is a partial order.

Definition 3.1.2 Let $\{i_j\}_{j=1}^{j=k}$ and $\{x_j\}_{j=1}^{j=k}$ be a two subsets of positive integers. Consider the shadows of points $P_j = (i_j,\, x_j)$ for j=1,...,k. Then the first shadow line $L_1(P_1, ..., P_n)$ is the (topological) boundary of the combined shadows of all the P_j that are not in the shadow of any other point.

<u>Note</u> :

1. The first shadow line is a same as boundary of the combined shadows of all the points P_j.
2. By abuse of notation we denote $L_1(P_1, ..., P_n)$ by L_1.
3. The points P_j which lie on L_1 are maximal w.r.t. $\trianglelefteq$.

The following theorem gives the description of first shadow line.

Theorem 3.1.3 Let $P_1, ..., P_n$ be as in definition 3.1.2. Then the first shadow line $L_1(P_1, ..., P_n)$ is a broken line consisting of line segments and exactly one horizontal and one vertical ray.

Proof: Observe that by suitable relabelling we may assume $P_1, ..., P_r$ are the set of all points on the first shadow line. Further we may assume $i_1 < i_2 < \cdots < i_r$. Then it follows from lemma 3.1 that $j_1 < j_2 < \cdots < j_r$. Then

L_1 = boundary of the combined shadows of $P_1, .., P_r$

= Boundary ($\cup_1^r$ shadow of P_k)

= Boundary ($\cup_1^r S_k$) where S_k = shadow of $P_k = \{(x, y)\colon\ x \geq i_k$ and $y \geq j_k\}$

$= (\cup_1^r S_k) \setminus (\cup_1^r S_k)^\circ$ where A° = interior of the set A.

$= (\cup_1^r S_k) \setminus (\cup_1^r T_k)$ where $T_k = \{(x, y)\colon\ x > i_k$ and $y > j_k\}$

$= \mathrm{V} \cup \mathrm{H}$

where $\mathrm{V} = \{(x, y)\colon\ x = i_1$ and $j_1 \leq y\} \cup \{(x, y)\colon\ x = i_k$ and $j_{k+1} \leq y \leq j_k$ for some $k = 1, .., r-1$ } and

$\mathrm{H} = \{(x, y)\colon\ y = j_r$ and $i_r \leq x\} \cup \{(x, y)\colon\ y = j_k$ and $i_k \leq y \leq i_{k+1}$ for some $k = 1, .., r-1$ }

Now the theorem follows immediately.

Corollary 3.1.4 The line $x = c$ intersects L_1 in a ray or a line segment if and only if there exists P_j whose x-coordinate is c. If above condition doesn't hold then the line $x = c$ intersects L_1 at a single point, say $(c,\ d)$ such that $(c \pm 1,\ d)$.

Proof: From the proof of previous theorem we have $L_1 = \mathrm{V} \cup \mathrm{H}$. The vertical line $x = c$ intersects L_1 in a ray or a line segment if and only if the line $x = c$ intersects V. The later case occurs if it intersects H but not V and proof follows from previous theorem.

Corollary 3.1.5 Let $(k,\ x)$, $(k + 1,\ y)$ be two points on the shadow line then $y \leq x$.

Proof: Follows immediately from theorem 3.1.3

Corollary 3.1.6 Suppose that $P_1, ..., P_r$ are the set of all points on the $L_1(P_1, ..., P_n)$. Then

1. x_{L_1} = the x-coordinate of L_i's vertical ray = min{x-coordinate of $P_1, ..., P_r$}
2. y_{L_1} = the y-coordinate of L_1's vertical ray = min{y-coordinate of $P_1, ..., P_r$}

Proof: We may assume $i_1 < i_2 < \cdots < i_r$ and $j_1 < j_2 < \cdots < j_r$. From theorem 3.1.3 , it follows the horizontal and vertical rays of L_1 are $\{(x, y)\colon\ y = j_r$ and $x \geq i_r\}$ and $\{(x, y)\colon\ x = i_1$ and $y \geq j_1\}$ respectively. Therefore $x_{L_1} = i_1 = \min\{i_1, ..., i_r\}$ and $y_{L_1} = j_r = \min\{j_1, ..., j_r\}$.

Definition 3.1.7 Given a set of points S=$\{(i_j, x_{i_j})\}_1^k$ in first quadrant, we form its shadow lines L_1,... as follows. Assuming that $L_1, .., L_{i-1}$ have been constructed, remove all points of S on these lines. Then L_i is the boundary of shadow of the remaining points of S.

Observe that L_j is first shadow line of points which are not on $L_1, .., L_{j-1}$. So, every shadow line is a broken line with one horizontal and one vertical ray. Therefore one can define x-coordinate and y-coordinate of L_j as

x_{L_j} = the x-coordinate of L_i's vertical ray,

y_{L_j} = the y-coordinate of L_i's horizontal ray

The shadow lines make up the *shadow diagram* .

Observe that intersection of any two shadow lines is empty and every point lies on a shadow line. Further the line $x = c$ intersects at most one of the L_j in a ray or a segment and rest all in single points. It follows from corollary 3.1.4 that the former condition holds if there exists a point P_j on L_j whose x-coordinate is c.

Lemma 3.1.8 Let $\{i_j\}_{j=1}^{j=k}$ and $\{x_j\}_{j=1}^{j=k}$ be a two subsets of positive integers. Consider the points $P_j = (i_j, x_j)$ for j=1,...,k. Let L_1 be denote first shadow of P_j . Then $x_{L_1} = \min\{i_j\}$ and $y_{L_1} = \min\{x_j\}$.

Proof: By suitable rearrangement we may assume that x_1 is the minimum of x_j's. Then (i_1, x_1) is not in shadow of no other point. Therefore (i_1, x_1) lies on L_1. Now lemma is an immediate consequence of corollary 3.1.6. The proof of other assertion is similar to the above.

Corollary 3.1.9 Let P_j be as in the previous lemma. Let $L_1, L_2,...$ be shadow lines in the shadow diagram of P_j. Then

$$x_{L_j} = \text{the minimum of the } x\text{-coordinate of } P_i\text{'s which are not on } L_1,...,L_{j-1}$$
$$y_{L_j} = \text{the minimum of the } y\text{-coordinate of } P_i\text{'s which are not on } L_1,...,L_{j-1}$$

Proof: This follows from immediately the observation that L_j is the first shadow line of points which are not on $L_1,..,L_{j-1}$ and corollary 3.1.6 .

Given $\pi \in S_n$, the *shadow diagram of* π is defined as the shadow diagram of points $(1,\pi(1))$, ... ,$(n,\pi(n))$.
To illustrate the properties of shadow line proved earlier we consider shadow diagrams of few permutations.
Example : Let $\pi = 4\ 1\ 3\ 5\ 6\ 2\ 7$

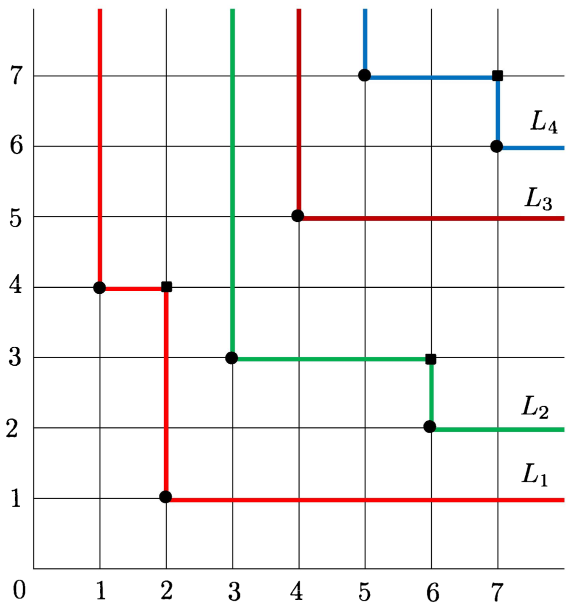

Figure 2: Shadow Diagram of π

§§ 3.2 R-S Correspondence using Shadow lines

In this section we describe R-S correspondence using the shadow lines. This remarkable geometric interpretation of R-S correspondence is made by Viennot. To motivate consider the figure 1. We observe the following :

	y- coordinate of lowest point of intersection			
$x = k$	L_1	L_2	L_3	L_4
$k = 1$	4	-	-	-
$k = 2$	1	-	-	-
$k = 3$	1	3	-	-
$k = 4$	1	3	5	-
$k = 5$	1	3	5	7
$k = 6$	1	2	5	7
$k = 7$	1	2	5	6

Table 1

From the table we observe the following remarkable property that 1^{st} row of $P_i(\pi_1)$ constructed in Schensted algorithm is the i^{th} row of the above. This indeed turns out be true in general !

Theorem 3.2.1 Let the shadow diagram of $\pi = x_1x_2...x_n$ constructed as before. Suppose the vertical line $x = k$ intersects i_k of the shadow lines. Let y_j be the y-coordinate of the lowest point of the intersection with L_j. Then the first row of the $P_k = \mathrm{P}(x_1, .., x_k)$ is $R_1 = y_1\ y_2\ ...\ y_{i_k}$

Proof: Induct on k, the lemma being trivial for $k = 0$ since there are no intersections. Assume that the result holds k and consider k+1. Since $y_1, y_2, ..., y_{i_k}$ form first row of P_k we have $y_1 < y_2 < \cdots < y_{i_k}$. So, there are two cases to consider

Case 1: $x_{k+1} > y_{i_k}$

By corollary 3.1.5, the point (k+1, x_{k+1}) doesn't lie on L_1,...,L_{i_k} as the y-coordinate of the lowest point of the intersection of $x = k$ with L_j is smaller than x_{k+1}. Therefore the point (k+1, x_{k+1}) starts a new shadow line. So none of the values $y_1, y_2, ..., y_{i_k}$ change, and we obtain a new intersection $y_{i_k+1} = x_{k+1}$. The number of shadow lines $x = k$+1 intersects is $i_{k+1} = i_k + 1$. Then first row of P_{k+1} is $y_1\ y_2\ ...\ y_i\ x_{k+1}$ which is same as y-coordinates of lowest point of the intersection $x = k + 1$ with $L_1, ..., L_{i_{k+1}}$.

Case 2: If there exists j such that, $y_1 < \cdots < y_j < x_{k+1} < y_j < \cdots < y_{i_k}$

The point (k+1, x_k) doesn't lie on $L_1, ..., L_j$ by case 1. If (k+1, x_{k+1}) lies on L_{j+r} for some $r > 1$, then L_{j+r} contains line segment $\{(x, y)$: $x = k$+1 and $x_{k+1} \leq y \leq y_{j+r}\}$. Then the point ($k$, y_j) lies on L_j and L_{j+r}. So, (k+1, x_{k+1}) get added to the line L_j and $y_i = x_{k+1}$. Other y_i's doesn't change. Schensted algorithm and induction hypothesis imply first row of P_{k+1} is $y_1...y_{j-1}x_{k+1}y_{j-1}...y_i$ which is same predicted by shadow diagram.

It follows from the proof of the previous lemma that the shadow diagram of π can be read left to right like a time-line recording the construction of P(π) [Table 1]. At the k^{th} stage, the line $x = k$ intersects one shadow line in a ray or line segment and all the rest in single points. In terms of the first row of P_k , a ray corresponds to placing an element at the end, a line segment corresponds to displacing an element, and the points correspond to elements that are unchanged.

Corollary 3.2.2 If the permutation π has Robinson-Schensted tableaux ($P(\pi)$, $Q(\pi)$) and shadow lines L_j, then for all j, $P_{1j}(\pi) = y_{L_j}$ and $\mathrm{Q}_{1j}(\pi) = x_{L_j}$.

Proof: The statement for P is just the case $k = n$ of Lemma 3.6 because $P_n = P$. As for $Q(\pi)$, the entry k appears $Q(\pi)$ in the cell (1,j) when x_k is greater than every element of the first row of $\mathrm{P}_{k-1}(\pi)$. But the previous lemma's proof shows every that this happens precisely when the line $x = k$ intersects shadow line L_j in vertical ray (Case-1 of previous lemma). In other words, $y_{L_j} = k = Q_{1j}(\pi)$ as desired.

The above corollary gives a method to construct the first row of $P(\pi)$ and $Q(\pi)$ from the shadow diagram of π. How do we recover the rest of the rows of P and Q tableaux from the shadow diagram of π ?

Definition 3.2.3 The north east (NE) corner of a shadow line such that $(x + 1, y)$ and $(x, y + 1)$ doesn't belong to the shadow line. These are indicated as by dots in the shadow diagram.

Note: From definition it follows NE corner is the rightmost / topmost point of horizontal / vertical segment respectively of a shadow line.

Now, consider the north-east corners of the shadow lines. If such a corner has coordinates (k, x'), then by the proof of Lemma 3.6 x' must be displaced from the first row of P_{k-1} by the insertion of x_k. For example, consider (3,5) in the diagram 2. From Schensted correspondence we observe 3 is bumped while inserting x_5= 2. So the NE corners correspond to the elements inserted into the later rows during the construction of P. Thus we can get the rest of the two tableaux by iterating the shadow diagram.

Definition 3.2.4 The i^{th} skeleton of $\pi \in S_n$ $\pi^{(i)}$ is defined inductively by $\pi^{(1)} = \pi$ and

$$\pi^{(i)} = \begin{matrix} k_1 & k_2 & \cdots & k_m \\ l_1 & l_2 & \cdots & l_m \end{matrix},$$

where $(k_1\ ,\ l_1), ..., (k_m\ ,\ l_m)$ are the north-east corners of the shadow diagram of $\pi^{(i-1)}$ listed in lexicographic order. The shadow lines for $\pi^{(i)}$ are denoted by $L_j^{(i)}$.
From the previous note it follows $\pi^{(i)}$ is a partial permutation.

We consider consider the example $\pi = 4\ 1\ 3\ 5\ 6\ 2\ 7$. From the first shadow diagram of π (Figure 2) we have $\pi^{(2)} = \begin{pmatrix} 2 & 6 & 7 \\ 4 & 3 & 7 \end{pmatrix}$.
The shadow diagram of $\pi^{(2)}$ is

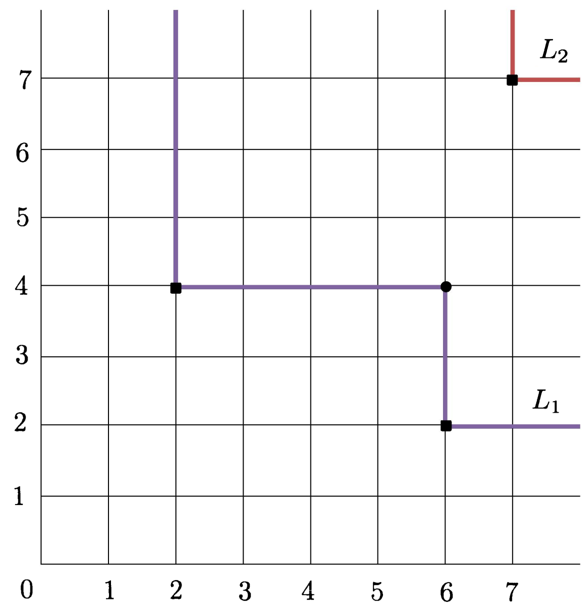

Figure 3: Shadow Diagram of $\pi^{(2)}$

We observe from above that $P_{21}(\pi) = 3 = x_{L_1(\pi^{(2)})}$ and $P_{22}(\pi) = 7 = x_{L_2(\pi^{(2)})}$. Similar observations can be made for $Q(\pi)$.
Theorem 3.2.5 ([Vie 76]) Suppose $\pi \xrightarrow{R-S} (P, Q)$. Then $\pi^{(i)}$ is a partial permutation such that $\pi^{(i)} \xrightarrow{R-S} (P^{(i)},\ Q^{(i)})$ where $P^{(i)}$(respectively $Q^{(i)}$) consists of the rows i and below of P (respectively, Q). Furthermore, $P_{ij} = y_{L_j^{(i)}}$ and $Q_{ij} = x_{L_j^{(i)}}$ for all i, j.
Proof: It is observed earlier that NE corners in shadow diagram of $\pi^{(i)}$ are the elements bumped from i^{th} row. Now the proof follows by statement analogous to corollary 3.2.2 (Theorem 3.2.1) for i^{th} shadow diagram. Since proof follows by similar arguments used in proving corollary 3.2.2 (Theorem 3.2.1) we skip the proof.

Theorem 3.2.6 ([Scü 63]) If $\pi \in S_n$, then P(π^{-1}) = Q(π) and Q(π^{-1}) = P(π) .
Proof: Taking the inverse of a permutation corresponds to reflecting the shadow diagram in the line $y = x$. The theorem now follows from Theorem 3.2.5

§4. Dual Knuth Relations

Now we consider the similar theme we carried out after R-S correspondence. Dual to our definition of P- equivalence is the following.

Definition 4.1 Two permutations π , $\sigma \in S_n$ are said to be Q -equivalent, written as $\pi \cong^Q \sigma$ if $Q(\pi) = Q(\sigma)$.

Example : 2 1 3 4 $\cong^Q$ 3 1 2 4 .

We also have a dual notion for the Knuth relations.

Definition 4.2 Two permutations π,$\sigma \in S_n$ are said to differ by a dual Knuth relation of the first kind, written $\pi \overset{1^\star}{\cong} \sigma$ if for some k,

1. $\pi = \ldots.. \ k+1 \ \ldots. \ k \ \ldots. \ k+2 \ . \ ..$ and $\sigma = \ldots.. \ k+1 \ \ldots. \ k \ \ldots. \ k+2 \ . \ ..$ or vice versa.

They differ by a dual Knuth relation of the second kind, written $\pi \overset{2^\star}{\cong} \sigma$, if for some k,

2. $\pi = \ldots.. \ k \ \ldots. \ k+2 \ \ldots. \ k+1 \ . \ ..$ and $\sigma = \ldots.. \ k+1 \ \ldots. \ k+2 \ \ldots. \ k \ . \ ..$ or vice versa.

The two permutations are dual Knuth equivalent, written as $\pi \overset{K^\star}{\cong} \sigma$ if there is a sequence of permutations such that $\pi \overset{i^\star}{\cong} \pi_1 \overset{j^\star}{\cong} \ldots.. \overset{l^\star}{\cong} \pi_k$ where $i,j,...,l \in \{1,2\}$.

Example: 2 1 3 4 $\overset{1^\star}{\cong}$ 3 1 2 4 and 4 2 3 1 $\overset{2^\star}{\cong}$ 4 1 3 2

Lemma 4.3 If π, $\sigma \in S_n$ then $\pi \overset{\mathrm{K}}{\cong} \sigma \iff \pi^{-1} \overset{\mathrm{K}^\star}{\cong} \sigma^{-1}$.

Proof: Observe that to prove the theorem it is enough to verify the following:

(i) $\pi \overset{1}{\cong} \sigma \iff \pi^{-1} \overset{1^\star}{\cong} \sigma^{-1}$ (ii) $\pi \overset{2}{\cong} \sigma \iff \pi^{-1} \overset{2^\star}{\cong} \sigma^{-1}$

We give of proof statement (i). It can be observed that proof of (ii) follows by similar arguments .

$\pi \overset{1}{\cong} \sigma \iff \exists\ k$ and $x < \pi(k+1) = y = \sigma(k+1) < z$ such that $\pi = \ldots. \ yxz \ \ldots.$ and $\sigma = \ldots. \ yzx \ \ldots.$ or vice versa $\iff \pi^{-1} = \ldots.. \ k+1 \ \ldots. \ k \ \ldots. \ k+2 \ . \ ..$ and $\sigma^{-1} = \ldots.. \ k+1 \ \ldots. \ k \ \ldots. \ k+2 \ . \ ..$ or vice versa $\iff \pi^{-1} \overset{1^\star}{\cong} \sigma^{-1}$

Theorem 4.4 If π, $\sigma \in S_n$ then $\pi \overset{\mathrm{K}^\star}{\cong} \sigma \iff \pi \overset{\mathrm{Q}}{\cong} \sigma$.

Proof:
$$\begin{aligned} \pi \overset{\mathrm{K}^\star}{\cong} \sigma &\iff \pi^{-1} \overset{\mathrm{K}}{\cong} \sigma^{-1} && (\text{ Lemma 4.3 }) \\ &\iff P(\pi^{-1}) = P(\sigma^{-1}) && (\text{ Thm 4.2 }) \\ &\iff Q(\pi) = Q(\sigma) && (\text{ Thm 3.2.6 }) \end{aligned}$$

From above theorem we have 2 1 3 4 $\overset{\mathrm{K}^\star}{\ncong}$ 4 2 3 1

References

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