The Murnaghan-Nakayama Rule

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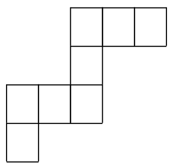
The Murnaghan-Nakayama rule gives us a combinatorial way of computing the character table of any symmetric group S_n.
Before illustrating this rule, we need to define a new class of objects called **Skew hook tableau**.

First we define what a skew hook is. **Skew hook(or rim hook)** is a skew diagram obtained by taking all cells on a finite lattice path with steps one unit northward or eastward.Equivalently,a skew hook is a skew diagram such that it is edgewise connected and contains no 2 x 2 subset of cells. A typical skew hook looks like:



If ζ is a skew hook, then we define the leg length of ζ(denoted by ll(ζ)) to be
(the number of rows in ζ) -1. In the above case the leg length is 3.

Skew hook tableau is a generalized tableau T with positive integral entries such that
i)rows and columns of T weakly increase
ii)all occurrences of i in T lie in a single rim hook.
Here is an example of a skew hook tableau.

1	2	2	3	3
2	2	3	3	
2	4	4	4	
2	4			
2				

Now we define the sign of a rim hook tableau with rim hooks $\xi^{(i)}$ to be

$$(-1)^T = \prod_{\xi^{(i)} \in T} (-1)^{\text{ll}(\xi^{(i)})}$$

The sign of the above rim hook tableau is 1.
Now we are in a position to state the result which will help us compute the character table of any symmetric group S_n.

Theorem 1: Let $\lambda \vdash n$ and let $\alpha = (\alpha_1, ..., \alpha_k)$ be any composition of n.Then

$$\chi^{\lambda}_{\alpha} = \sum_{T} (-1)^{T}$$

where the sum is over all rim hook tableaux of shape λ and content α.

Remark: From Theorem 1, we can see that all the irreducible characters of S_n are integer valued.

Let us consider an example:
Suppose $\lambda = (3, 2)$ and $\upsilon = (2, 2, 1)$. Then all the rim hook tableaux are:

1	1	3
2	2	

,

1	2	3
1	2	

,

1	2	2
1	3	

The signs of the rim hooks are $1, 1$ and -1 respectively. So,

$$\chi^{(3,2)}_{(2,2,1)} = 1 + 1 - 1 = 1$$

Now,we will prove the following statement, from which theorem 1 will follow naturally.

Theorem 2: If $\lambda \vdash n$ and $\alpha = (\alpha_1, ..., \alpha_k)$ is a composition of n, then

$$\chi^{\lambda}_{\alpha} = \sum_{\xi} (-1)^{ll(\xi)} \chi^{\lambda \setminus \xi}_{\alpha \setminus \alpha_1}$$

where the sum runs over all rim hooks ξ of λ having α_1 cells.

Note that there may exist some rim hook ξ such that $\lambda \setminus \xi$ may not be a proper shape or all the cells may be deleted. for e.g.

	•	
•	•	

or,

•
•

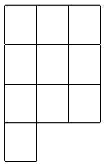
Here, the dotted cells represent the hook.

In the first case we set $\chi^{\lambda \setminus \xi}_{\alpha \setminus \alpha_1}$ to be 0 and for the second case set $\chi^{(0)}_{(0)} = 1$.

Example: For large n, it might not be easy to find the rim hook tableaux. In that case, theorem 2 can be used.
Let $\lambda = (7, 4, 3, 1)$ and $\alpha = (5, 4, 3, 2, 1)$. Then, to apply theorem 2, we need to find the rim hooks ξ with 5 cells in λ such that $\lambda \setminus \xi$ has proper shape. We can see that there is only one such hook:

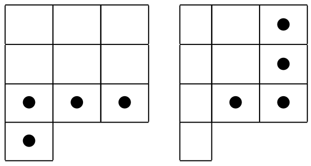
			•	•	•	•
			•			

Here,the rim hooks to be removed are denoted by dots. So, after removing we get:

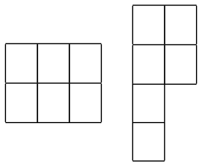


$$\chi^{(7,4,3,1)}_{(5,4,3,2,1)} = -\chi^{(3,3,3,1)}_{(4,3,2,1)}$$

There are two rim hooks with 4 cells within the above figure:

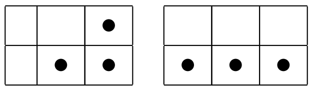


So, after removing we get,

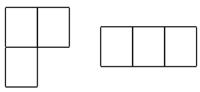


$$\chi^{(7,4,3,1)}_{(5,4,3,2,1)} = -\chi^{(3,3,3,1)}_{(4,3,2,1)} = -(-\chi^{(3,3)}_{(3,2,1)} + \chi^{(2,2,1,1)}_{(3,2,1)})$$

Now, there are no rim hooks with 3 cells within the second figure, and two rim hooks with 3 cells in the first figure:

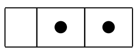


So, after removing the rim hooks they become

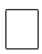


$$\chi^{(7,4,3,1)}_{(5,4,3,2,1)} = -\chi^{(3,3,3,1)}_{(4,3,2,1)} = -(-\chi^{(3,3)}_{(3,2,1)} + \chi^{(2,2,1,1)}_{(3,2,1)}) = -(-(\chi^{(3)}_{(2,1)} - \chi^{(2,1)}_{(2,1)}) + 0)$$

There are no rim hooks with 2 cells in the first figure and in the second figure there is only one rim hook with 2 cells:



So, after removing we get,



So,

$$\chi^{(7,4,3,1)}_{(5,4,3,2,1)} = -(-(\chi^{(3)}_{(2,1)})) = -(-(\chi^{(1)}_{(1)})) = 1$$

Proof of Theorem 1 from Theorem 2: We will proceed by induction on k. The case $m = 1$ follows from Lemma 1, which will be proved later.

Suppose the statement is true for $k-1$. Then, by theorem 2 and induction hypothesis,

$$\chi^{\lambda}_{(\alpha_1,\alpha_2,\ldots,\alpha_k)} = \chi^{\lambda}_{(\alpha_k,\alpha_2,\ldots,\alpha_1)} = \sum_{\xi}(-1)^{\text{ll}(\xi)}\chi^{\lambda\backslash\xi}_{\alpha\backslash\alpha_k} = \sum_{\xi}(-1)^{\text{ll}(\xi)}\sum_{T_\xi}(-1)^{T_\xi}$$

where ξ is all rim hooks of λ with α_k cells such that $\lambda \setminus \xi$ has proper shape and T_ξ is a rim hook tableau of shape $\lambda \setminus \xi$ and content $\alpha \setminus \alpha_k$.(to avoid too many new names, we, for now, denote $(\alpha_k, ..., \alpha_1)$ by α.)

Now it is easy to see that the sets $\{(\xi, T_\xi)|$ ξ is all rim hooks of λ with α_k cells such that $\lambda \setminus \xi$ has proper shape and T_ξ is a rim hook tableau of shape $\lambda \setminus \xi$ and content $\alpha \setminus \alpha_k$ $\}$ and $\{$ T $|$ T is a rim hook tableau of shape λ and content α $\}$ are bijective.
(For a given rim hook tableau of shape λ and content α, the hook corresponding to the maximum element which appear in T will be a hook with α_k cells and removing the hook from T will give a rim hook tableau of shape $\lambda \setminus \xi$ and $\alpha \setminus \alpha_k$. And for a pair (ξ, T_ξ) construct a rim hook tableau of shape λ and content α by placing the tableau T_ξ in the place $\lambda \setminus \xi$ within λ and placing the rim hook ξ in the place ξ within λ.)
Also , if we denote this bijection by f , then

$$(-)^{f((\xi, T_\xi))} = (-1)^{\mathrm{ll}(\xi)}(-1)^{T_\xi}$$

So, it follows that

$$\chi^\lambda_\alpha = \sum_T (-1)^T$$

where T runs over all rim hook tableaux of shape λ and content α.

Proof of Theorem 2: Let $m = \alpha_1$. Consider $\pi\sigma \in S_{n-m} \times S_m \subseteq S_n$,where $\pi \in S_{n-m}$ has type $(\alpha_2, ..., \alpha_k)$ and $\sigma \in S_m$ is a m-cycle. So, $\pi\sigma$ has type $(\alpha_1, \alpha_2, ..., \alpha_k)$.

$$\text{Therefore, } \chi^\lambda_\alpha = \chi^\lambda(\pi\sigma) = \chi^\lambda \downarrow_{S_{n-m} \times S_m} (\pi\sigma)$$

Now, since $\{\chi^\mu | \mu \vdash n - m\}$ and $\{\chi^\upsilon | \upsilon \vdash m\}$ are the sets of all irreducible characters of S_{n-m} and S_m respectively, so $\{\chi^\mu \otimes \chi^\upsilon | \mu \vdash n - m, \upsilon \vdash m\}$ is the set of all irreducible characters of $S_{n-m} \times S_m$.

$$\chi^\lambda_\alpha = \chi^\lambda(\pi\sigma) = \chi^\lambda \downarrow_{S_{n-m} \times S_m} (\pi\sigma) = \sum_{\substack{\mu \vdash n-m \\ \upsilon \vdash m}} m^\lambda_{\mu\upsilon} \chi^\mu \otimes \chi^\upsilon(\pi\sigma) = \sum_{\substack{\mu \vdash n-m \\ \upsilon \vdash m}} m^\lambda_{\mu\upsilon} \chi^\mu(\pi)\chi^\upsilon(\sigma)$$

$$\text{where } m^\lambda_{\mu\upsilon} = \langle \chi^\lambda \downarrow_{S_{n-m} \times S_m}, \chi^\mu \otimes \chi^\upsilon \rangle$$

Then using Frobenius reciprocity,we get

$$m^\lambda_{\mu\upsilon} = \langle \chi^\lambda, (\chi^\mu \otimes \chi^\upsilon) \uparrow^{S_n} \rangle$$

Now, we know that $\langle \chi^\lambda, (\chi^\mu \otimes \chi^\upsilon) \uparrow^{S_n} \rangle$ is the Littlewood-Richardson coefficient, denoted by $c^\lambda_{\mu\upsilon}$

So, the equation becomes

$$\chi^\lambda(\pi\sigma) = \sum_{\mu \vdash n-m} \chi^\mu(\pi) \sum_{\upsilon \vdash m} c^\lambda_{\mu\upsilon} \chi^\upsilon(\sigma) \tag{1}$$

Now we will evaluate $\chi^\upsilon(\sigma)$, where σ is a m-cycle.

Lemma 1:If $\upsilon \vdash m$ then,

$$\chi^{\upsilon}_{(m)} = \begin{cases} (-1)^{m-r} & \text{if } \upsilon = (r, 1^{m-r}) \\ 0 & otherwise. \end{cases}$$

Proof: We know that

$$s_\upsilon = \sum_\mu \frac{1}{z_\mu} \chi^\upsilon_\mu p_\mu$$

where s_υ is the schur function associated with the partition λ , z_μ is the order of the centralizer of any element of type μ and p_μ is the power sum function associated with μ.
So,$\chi^\upsilon_{(m)}$ is $z_{(m)}$ times the coefficient of $p_{(m)}$.
Now, by the the complete homogeneous Jacobi-Trudi determinant, we get

$$s_\upsilon = ||h_{\upsilon_i - i + j}||_{l \times l} = \sum_\kappa \pm h_\kappa$$

.
where h_κ is the complete homogeneous symmetric function associated with κ, and the sum is over all compositions $\kappa = (\kappa_1, ..., \kappa_l)$ that occur as a term in the determinant. Now ,since $\{p_{\alpha_i} | \alpha_i \vdash \kappa_i\}$ forms a basis of Λ^{κ_i} , so

$$h_\kappa = \prod_i (\sum_{\alpha_i \vdash \kappa_i} a_{\alpha_i} p_{\alpha_i})$$

.

If $p_{(m)}$ occurs in this sum, then for some $\alpha_1 \vdash \kappa_1, \alpha_2 \vdash \kappa_2, ..., \alpha_l \vdash \kappa_l$ we get

$$a p_{\alpha_1} p_{\alpha_2} .... p_{\alpha_l} = p_{(m)}.$$

where a is a constant.

So,some α_i must be m and so, $\kappa_i = m$. $\therefore$ $\chi^\upsilon_{(m)} \neq 0$ only when h_m appears in the preceding determinant.

The largest index to appear in this determinant is at the end of the first row, and $\upsilon_1 - 1 + l = h_{1,1}$, the hook length of $(1,1)$ in a tableau of shape υ. So, we always have $m = |\upsilon| \geq h_{1,1}$. Thus$\chi^\upsilon_{(m)} \neq 0$ only when $h_{1,1} = m$ i.e., when υ is a hook $(r, 1^{m-r})$.In this case, we have,

$$s_\upsilon = \begin{Vmatrix} h_r & ..... & & & h_m \\ h_0 & h_1 & ..... & & \\ 0 & h_0 & h_1 & ..... & \\ 0 & 0 & h_0 & h_1 & ..... \\ . & . & . & & \\ . & . & . & & \\ . & . & . & & \end{Vmatrix}$$

$$= (-1)^{m-r} h_m$$

$$+ \text{ other terms not involving } p_m$$

Now, we know that $h_m = s_{(m)}$ and the coefficient of p_m in s_m is given by $\frac{1}{m}\chi^{(m)}_{(m)}$, which is equal to $1/m$. So,

$$\chi^\upsilon_{(m)} = (-1)^{m-r}$$

□

Lemma 2:Let $\lambda \vdash n, \mu \vdash n-m$ and $\upsilon = (r, 1^{m-r})$. Then $c^{\lambda}_{\mu\upsilon} = 0$ unless each edgewise connected component of of $\lambda \setminus \mu$ is a rim hook. In that case, if there are k component hooks spanning a total of c columns, then

$$c^{\lambda}_{\mu\upsilon} = \binom{k-1}{c-r}$$

.

Proof: By the Littlewood-Richardson rule, $c^{\lambda}_{\mu\upsilon}$ = number of semi-standard tableaux T of shape $\lambda \setminus \mu$ with content $\upsilon = (r, 1^{m-r})$ such that π_T is a reverse lattice permutation.i.e.if $\pi_1, \pi_2, ..., \pi_l$ are the rows of T, then the sequence $\pi^r_T = \pi^r_1 \pi^r_2 .... \pi^r_l$ is a lattice permutation.Note that since the content of of T is $(r, 1^{m-r})$, so there are exactly r 1's in T and the numbers 2,3,...,$(m-r+1)$appears exactly once in T.From this we can see that the numbers greater than 1 appears in increasing order in π^r_T.This condition together with semi-standardness puts the following constraints on T:

T1:Any cell of T having a cell to its right must contain a 1. (If it contains $s > 1$ then, cell to its right must contain a number $q \geq s > 1$ because of semi-standardness. But in π^r_T, q appears before s, so $q < s$.Now we can see that this is not possible.)

T2:Any cell of T having a cell above must contain an element bigger than 1.(in a semi-standard tableau columns are strictly increasing.)

From T1 and T2, it follows that if T contains a 2×2 square then there is no way of filling the lower left cell, so $c^{\lambda}_{\mu\upsilon} = 0$ if any one of the components is not a rim hook.

Now suppose that $\lambda \setminus \mu = \biguplus_{i=1}^{k} \xi^{(i)}$, where each $\xi^{(i)}$ is a rim hook. T1, T2 and the fact that the numbers 2 through $m-r+1$ increase in π^r_T show that $\xi^{(i)}$ is of the form

		1	1	1	b
		d			
1	1	d+1			
d+2					
d+3					

Here we order the $\xi^{(i)}$ such that the number of the first row of $\xi^{(i)}$ is less than the number of the first row of $\xi^{(i+1)}$.Then the $d > 1$ is the smallest number that has not appeared in $\pi^r_{\xi^{(1)}} \pi^r_{\xi^{(2)}} .... \pi^r_{\xi^{(i-1)}}$ and b is either 1 or $d-1$.Also, in $\xi^{(1)}$, b=1(the first element in$\pi^r_{\xi^{(1)}}$ is this b).Now, by T1 and T2 we get that the number of 1's fixed in $\xi^{(1)}$ is the number of columns of $\xi^{(1)}$ and for any $i > 1$, number of 1's fixed in $\xi^{(i)}$ is number of columns of $\xi^{(i)} - 1$.So, number of 1's fixed in T is $c-k+1$(number of columns of T = $\sum_i$ number of columns of $\xi^{(i)}$ since any two distinct component hooks cannot have a common column).Hence there are $r-c+k-1$ 1's left to distribute among the $(k-1)$ cells marked with a b. The number of ways this can be done is

$$c^{\lambda}_{\mu\upsilon} = \binom{k-1}{r-c+k-1} = \binom{k-1}{r-c}.$$

□

Putting the values from lemma 1 and 2 in equation 1, we get

$$\chi^{\lambda}(\pi\sigma) = \sum_{\mu \vdash n-m} \chi^{\mu}(\pi) \sum_{\upsilon \vdash m} c^{\lambda}_{\mu\upsilon} \chi^{\upsilon}(\sigma) = \sum_{\mu \vdash n-m} \chi^{\mu}(\pi) \sum_{r=1}^{m} \binom{k-1}{c-r} (-1)^{m-r} \tag{2}$$

Now, $k \leqslant c \leqslant m$, since k is the number of skew hooks $\xi^{(i)}$, c is the number of columns in all the $\xi^{(i)}$, and m is the number of cells in all the $\xi^{(i)}$. So

$$\sum_{\mu \vdash n-m} \chi^{\mu}(\pi) \sum_{r=1}^{m} \binom{k-1}{c-r} (-1)^{m-r} = \sum_{\mu \vdash n-m} \chi^{\mu}(\pi) \sum_{r=0}^{k-1} \binom{k-1}{r} (-1)^{m-r}$$

$$= (-1)^{m-c} \sum_{\mu \vdash n-m} \chi^{\mu}(\pi) \sum_{r=0}^{k-1} \binom{k-1}{r} (-1)^{c-r} = \begin{cases} (-1)^{m-c} & \text{if } k-1=0 \\ 0 & \text{otherwise} \end{cases}$$

But if $k = 1$, $\lambda \setminus \mu$ is a single rim hook ξ with m cells and c columns.Hence $m - c = \mathrm{ll}(\xi)$, so equation 2 becomes

$$\chi^{\lambda}(\pi\sigma) = \sum_{|\xi|=m} (-1)^{\mathrm{ll}(\xi)} \chi^{\lambda\setminus\xi}(\pi)$$

□