10/02/2013

Representations of symmetric groups

Homework 6

(Due on 17/02/2014 at 9:10 a.m.)

Instructions:

- Solutions must be complete and legible in order to earn maximum points.
- You may discuss and work together if necessary but you must **write your own solutions**. Copied solutions (from each other or books or the internet) are easy to identify and easier to grade as they can only earn a zero.

1. Let $n = 4$. Answer each of the following questions for the partitions $\lambda \in \{(2,2),(2,1,1)\}$.

   (a) List a basis of tabloids for M^λ.

   (b) Find the induced representation $1 \uparrow_{S_\lambda}^{S_4}$ by writing out the cosets explicitly. Find an isomorphism of S_4-modules $\theta_\lambda : 1 \uparrow_{S_\lambda}^{S_4} \to M^\lambda$.

   (c) Let $t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$ (resp. $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$) be a λ-tableau. What are the corresponding R_t, C_t, κ_t and e_t?

   (d) Let $s = \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & 4 \\ \hline \end{array}$ (resp. $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$) be another λ-tableau. Find $\sigma \in S_\lambda$ such that $t = \sigma s$. What is the relation between e_s and e_t?

   (e) What is $\kappa_t\{s\}$?

2. Let G be a group and $H \subset G$ have index 2. Prove the following:

   (a) H is normal in G.

   (b) Every conjugacy class of G having non-empty intersection with H becomes a conjugacy class of H or splits into two conjugacy classes of H of equal size. Furthermore, the conjugacy class K of G does not split in H if and only if some $k \in K$ commutes with some $g \notin H$.

   (c) Let χ be an irreducible character of G. Then $\chi \downarrow_H$ is either irreducible or is the sum of two inequivalent irreducibles. Furthermore, $\chi \downarrow_H$ is irreducible if and only if $\chi(g) \neq 0$ for some $g \notin H$.

3. Let A_n denote the alternating subgroup of S_n and consider $\pi \in S_n$ having cycle type $\lambda = (\lambda_1, \ldots, \lambda_l)$.

   (a) Show that $\pi \in A_n$ if and only if $n - l$ is even.

   (b) Prove that the conjugacy classes of S_n that split in A_n are those where all parts of λ are odd and distinct.

   Observe that you can use the above two exercises to find the character table of A_4 using the character table of S_4.