13/01/2014

Representations of the symmetric group

Homework 2

(Due on 20/01/2014 at 9:10 a.m.)

Instructions:

- Solutions must be complete and legible in order to earn maximum points.
- You may discuss and work together if necessary but you must write your own solutions. Copied solutions (from each other or books or the internet) are easy to identify and easier to grade as they can only earn a zero.

1. Let G act on S with the corresponding permutation representation. Prove the following:
   (a) The matrices for the action of G in the standard basis are permutation matrices.
   (b) If the character if this representation is χ and $g \in G$, then
   $$\chi(g)=\text{number of fixed points of } g.$$
2. Let G be an abelian group. Find all inequivalent irreducible representations of G. (Hint: Use the fundamental theorem of abelian groups).
3. Prove Lemma 0.2, Theorem 0.3 and Theorem 0.4 in the following note.

   Here is another way one can prove the complete reducibility theorem when the vector space V is equipped with a hermitian inner product $\langle\ ,\ \rangle$.

   Definition 0.1 *We say that a representation $\rho : G \to \mathrm{GL}(V)$ is* ***unitary*** *if the inner product on V is preserved by the action of G i.e. for all $s \in G$ and $v, w \in V$,*
   $$\langle \rho_s(v), \rho_s(w)\rangle = \langle v, w\rangle.$$

   In particular, a unitary G-action preserves orthonormal bases. A representation is unitary iff the image of ρ is contained in the subgroup of unitary matrices in $\mathrm{GL}(V)$. The following lemma is the unitary version of Theorem 1.1 we did in class.

   Lemma 0.2 *Let V be a unitary representation of G and W be a G-invariant subspace of V. Then the orthogonal complement $W^{\perp}$ is also G-invariant.*

   Theorem 0.3 *Any finite-dimensional unitary representation of a group admits an orthogonal decomposition into irreducible unitary sub-representations.*

   The next theorem says that any finite-dimensional representation of $\rho : G \to \mathrm{GL}(V)$ can be made unitary i.e. any hermitian inner product on V can be suitably 'averaged' to form an inner product that is preserved by G.

   Theorem 0.4 *(Weyl's unitary trick) Finite-dimensional representations of finite groups are unitarisable.*

   Hint: Construct a new hermitian inner product
   $$\langle v, w\rangle' = \frac{1}{|G|}\sum_{s\in G}\langle \rho_s(v), \rho_s(w)\rangle.$$

   Check that this inner product gives us the required result.

   Conclude that Theorem 0.3 and Theorem 0.4 together imply the complete reducibility theorem.