24/03/2014

Representations of the symmetric group

Homework 10

(Due on 31/03/2014 at 9:10 a.m.)

Instructions:

- Solutions must be complete and legible in order to earn maximum points.
- You may discuss and work together if necessary but you must write your own solutions. Copied solutions (from each other or books or the internet) are easy to identify and easier to grade as they can only earn a zero.

1. Find the matrix and the generalized permutation which corresponds to the pair

$$P = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}\,, \qquad Q = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}$$

under the R-S-K correspondence.

2. (Properties of sliding) Let $\lambda = (4,4,3,2)$, $\mu = (2,1)$.

   (a) Let P be the partial skew tableau $\begin{array}{cccc} \cline{3-4} & & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\ \cline{2-4} & \multicolumn{1}{|c|}{3} & 5 & \multicolumn{1}{c|}{5} \\ \cline{1-4} \multicolumn{1}{|c|}{4} & 4 & \multicolumn{1}{c|}{6} & \\ \cline{1-3} \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{6} & & \\ \cline{1-2}\end{array}$ of shape $\lambda \setminus \mu$. Let the cells of μ be named as $\begin{array}{|c|c|}\hline x & y \\ \hline z \\ \cline{1-1}\end{array}$.

   Apply the forward slide j^c to the inner corner $c = z$ of μ.

   (b) Suppose in part (a), the algorithm ends at a cell d and the resulting tableau is denoted by $j^c(P)$. Apply the backward slide j_d to $j^c(P)$ and find the resulting tableau.

   (c) Obtain a normal tableau from P in two ways, by applying the sliding sequences zyx and yzx respectively. Do you get the same normal tableau?

3. For the partial skew tableau P in the previous example, show that the row word $w_{row}(P)$ is Knuth equivalent the column word $w_{col}(P)$. (One reads a column word bottom to top starting at the leftmost column and moving right.)

4. For $\lambda = (4,4,3,2)$, calculate f^λ using

   (a) the hook formula;

   (b) the determinantal formula.

   How does the determinant in part (b) relate to the determinant obtained for $\overline{\lambda} = \lambda \setminus \{\text{first column}\} = (3,3,2,1)$.

5. Express the product $m_{(2,1)} m_{(1,1)}$ as a linear combination of monomial symmetric functions.

   In general, given partitions λ and μ, show that the coefficient of m_ν in the expansion of $m_\lambda m_\mu$ in terms of monomial symmetric functions is the number of ways of expressing the multi-index $\nu = (\nu_1, \ldots, \nu_n, 0, \ldots)$ as a sum $\alpha + \beta$ of multi-indices, where α has shape λ and β has shape μ.

   (A multi-index is an infinite sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ of non-negative integers of which only finitely many are non-zero. For each multi-index α, $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ is a monomial of degree $|\alpha| = \alpha_1 + \alpha_2 + \cdots$.

   Given a multi-index α, its positive terms can be arranged in weakly decreasing order to obtain a partition, which is called the *shape of* α. For example, if $\alpha = (0,2,0,0,3,1,5,0,0,\ldots)$, then $x^\alpha = x_2^2 x_5^3 x_6 x_7^5$ and the shape of α is $(5,3,2,1)$.)