06/01/2013

Representations of the symmetric group

Homework 1

(Due on 13/01/2014 at 9:10 a.m.)

Instructions:

- Solutions must be complete and legible in order to earn maximum points.
- You may discuss and work together if necessary but you must **write your own solutions**. Copied solutions (from each other or books or the internet) are easy to identify and easier to grade as they can only earn a zero.

1. If a group G acts on a set S and $s \in S$, then the *stabilizer of* s is $G_s = \{g \in G \mid gs = s\}$ and the *orbit of* s is $\mathcal{O}_s = \{gs \mid g \in G\}$.
   (a) Prove that G_s is a subgroup of G.
   (b) Find a bijection between the cosets of G/G_s and the elements of $\mathcal{O}_s$.
   (c) Show that $|\mathcal{O}_s| = |G|/|G_s|$.
2. Let V be a finite-dimensional vector space and W be a subspace of V. Prove that there is a bijective correspondence between the projections of V onto W and the complements of W in V.
3. Let V be a 4-dimensional vector space with basis $\{e_1, e_2, e_3, e_4\}$. List the corresponding bases of $V \otimes V$, $\mathrm{Sym}^2 V$ and $\bigwedge^2 V$.
4. Let V be a representation of G containing a vector v such that the set $\{\rho_s(v)\}_{s\in G}$ forms a basis of V. Prove that V is isomorphic to the regular representation of G.

   (Here $\rho : G \to \mathrm{GL}(V)$ is the representation of G in V and ρ_s stands for the element $\rho(s) \in \mathrm{GL}(V)$.)
5. Let V be the regular representation of G and W be the 1-dimensional subspace of V generated by the element $x = \sum_{s\in G} e_s$. Prove that W is a subrepresentation of V and that W is isomorphic to the trivial representation of G.