02/01/2017

MSc. Applications of Mathematics

Linear Algebra - Homework 1

(Due on 09/01/2017 at 10:30 a.m.)

Instructions:

- Solutions must be complete and legible in order to earn maximum points.
- You may discuss and work together if necessary but you must write your own solutions. Copied solutions (from each other or books or the internet) are easy to identify and easier to grade as they can only earn a zero.

1. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by:

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 + x_3 + 2x_4 \\ x_1 - 2x_2 + x_3 - x_4 \\ 2x_1 - 5x_2 + x_4 \end{bmatrix}$$

   (i) What is the matrix representing T in the standard bases?

   (ii) What happens if the basis of $\mathbb{R}^4$ is changed to $\mathcal{B} = \left\{ v_1 = \begin{bmatrix} 7 \\ 3 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -4 \\ -2 \\ 0 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 7 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$?

   (iii) What happens if the basis of $\mathbb{R}^3$ (only) is changed to $\mathcal{B}' = \left\{ w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, w_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$?

   (iv) What happens if both bases are changed simultaneously?

   (v) Express the answers to parts (ii), (iii) and (iv) in terms of multiplying T (on the right or left or both) by an appropriate change of basis matrix.

2. Let $A = \begin{pmatrix} 3 & -2 & -2 \\ 1 & 0 & -2 \\ 3 & -3 & -1 \end{pmatrix}$

   (i) What is the characteristic polynomial of A? What is its minimal polynomial?

   (ii) Is the matrix diagonalizable? If yes, write the matrix in diagonal form. Is this form unique?

   (iii) Write the matrix P which diagonalises A.

3. Find the Jordan canonical form of the following matrix:

$$A = \begin{pmatrix} -6 & 1 & 3 \\ 11 & -1 & -5 \\ -24 & 3 & 11 \end{pmatrix}$$

   What are the algebraic and geometric multiplicities of each eigenvalue?

4. Write all possible Jordan canonical forms if the characteristic polynomial is $(x-2)^4(x-3)^2$.

5. Let A be a Hermitian matrix.

   (i) Prove that all eigenvalues of A are real.

   (ii) Prove that if x and y are eigenvectors corresponding to distinct eigenvalues then x and y are orthogonal.