13/08/2013

MSc. Applications of Mathematics

Linear Algebra - Homework 2

(Due on 19/08/2013 at 10:30 a.m.)

Instructions:

- Solutions must be complete and legible in order to earn maximum points.
- You may discuss and work together if necessary but you must write your own solutions. Copied solutions (from each other or books or the internet) are easy to identify and easier to grade as they can only earn a zero.

1. (Frobenius norm: Example of a non-subordinate matrix norm) Let A be an $m \times n$ matrix. Define

$$||A||_F = (\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2)^{\frac{1}{2}}.$$

   (i) Prove that this is indeed a matrix norm.

   (ii) Further prove that $||A||_F = \sqrt{\text{tr}A^*A} = \sqrt{\text{tr}AA^*}$ where $\text{tr}(B)$ denotes the sum of its diagonal entries.

   (iii) Evaluate the Frobenius matrix norm for

$$\begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}.$$

2. Find the 1, 2 and ∞-norms of $x = [2,\ 1,\ -4,\ -2]^t$ and $x = [1+i,\ 1-i,\ 1,\ 4i]^t$.

3. Evaluate the induced 1, 2 and ∞-matrix-norms for the matrix in exercise 1(iii).

4. Explain why $||I_{n\times n}|| = 1$ for every induced matrix norm. What is $||I_{n\times n}||_F$?

5. If $x \in \mathbb{C}^m$ and A is a $m \times n$ matrix, prove the following inequalities:

   (a) $||x||_\infty \leq ||x||_2$.

   (b) $||x||_2 \leq \sqrt{m}||x||_\infty$.

   (c) $||A||_\infty \leq \sqrt{n}||A||_2$.

   (d) $||A||_2 \leq \sqrt{m}||A||_\infty$.