Integer Programming Explained Through Gomory's Cutting Plane Algorithm and Column Generation

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Proposal

Description : ILP [integer linear programming] is an extension of Linear Programming, with an additional restriction that the variables should be integer valued. A standard ILP is of the form:

- An ILP differs from the LP in the context of feasible region. We shall give an intro of polyhedral theory, defining what is meant by polytope, convex hull and a few other relevant ideas.
- **Motivation :** We will state why ILP is necessary in today's world. We will illustrate the problems we face with classical LP, that is when the strict imposition of integral solutions is ignored.We use the knapsack problem to detail.
- **Formulation :** We show how to formulate ILP using examples.
- **Methods :** Classification into 2 broad divisions ,namely exact and non-exact . We will focus on 2 specific methods of exact types ,namely the Gomory's Cutting Plane Algoriithm and Column Generation. We also give a brief on relaxation .
  - Gomory's Cutting Plane : Construction of Gomory Cuts ,generation of gomory constraints ,general algorithm for solving ILP, Cutting point Algorithm.

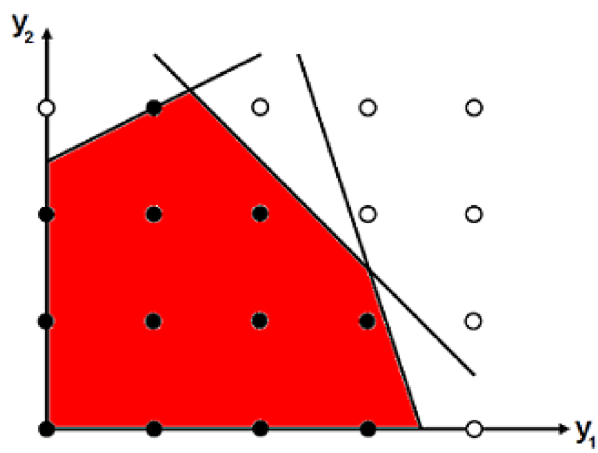
- Column Generation : A particular type of decomposition technique. We explain the intricate details of this method via VRP[vehicle routing problem]. Generic Column Generation Algorithm will be provided. [We will probably talk about Dantzig-Wolfe Decomposition via air-crew scheduling]

- **Outcome :** We will explain how the problems stated initially will be solved ,how graphical methods are used to solve complex situations and present a comparative study of the complexities of the 2 methods we study .

Report 1

In this report, we state why ILP is indispensable. Suppose we want to find the optimal allocation of labor to various parts of an industry with certain constraints which leads to an optimal solution where the solution vector is fractional. But labor quantities cannot be fractional, and hence, the need of ILP.

- An ILP differs from LP in the basic context of feasible region. In an LP the feasible region is basically in convex set in some finite dimension, but in an ILP the feasible sets are nothing but disjoint unions of discrete points. Here is a graphical representation:



Feasible regions of LP and ILP.

The black circles denote the of ILP solutions, while the shaded region represents the feasible region of the LP.

- We have found a general algorithm for solving a system of linear equations by *Simplex Method*.

- We are working on *Knapsack Problem* and *Zero-One IP*. We shall use this to demonstrate the technique of formulation of an ILP.

- Then we talk about the use of relaxation theory, *i.e.* when the strict condition of integral vector is not imposed. The associated programming problem is *Linear Relaxation LR*. If the objective function coefficients have integer values, then for minimization, the optimal objective for the ILP is greater than or equal to the rounded-off value of the optimal for LR. For maximization, the optimal objective for the ILP is less than or equal to the rounded-off value of the optimal objective for LR.

For a minimization ILP, the optimal objective value for LR is less than or equal to the optimal objective for ILP and for a maximization ILP, the optimal objective value for LR is greater than or equal to that of ILP. If LR is infeasible, then ILP is also infeasible. Also, if LR is optimized by integer variables, then that solution is feasible and optimal for ILP.

- We have explored the various details about Gomory Cut. The details are briefed below: Assuming x to be integral and non-negative, we construct Gomory Cut for $Ax = b$. We have found out how to generate cuts, *i.e.*finding the additional constraints which generates new hyperplanes which refines the feasible region.We have also learnt about the different aspects of Gomory constraints, such as logical constraints, alternative constraints and conditional constraints.

  We are working on finding out the complexity of the method using a generic algorithm, which we have already obtained.

Report 2

We have explored the geometrical aspects which are associated with ILP, such as the meanings of polytope, convex hull, convex sets and their unions or intersections and how these are used in ILP.

In this report we mainly focused on the Column Generation and why it is essential to solve a bit complicated ILPs. The cutting plane method which we explained earlier is theoretically perfect to solve any type of ILP but practically when it comes solving ILPs involving large number of variables, the cutting plane technique would be very cumbersome, it is when column generation comes into the picture.

Column generation is a method for solving ILPs without having to enumerate all the unknown variables a priori.

We take the *cutting stock problem* and *pricing problem* as examples to illustrate this technique. We first define the symbols, then formulate the master ILP and then work on the sub problem of finding the minimum cost solution.

We are trying to find a generic column generation algorithm.