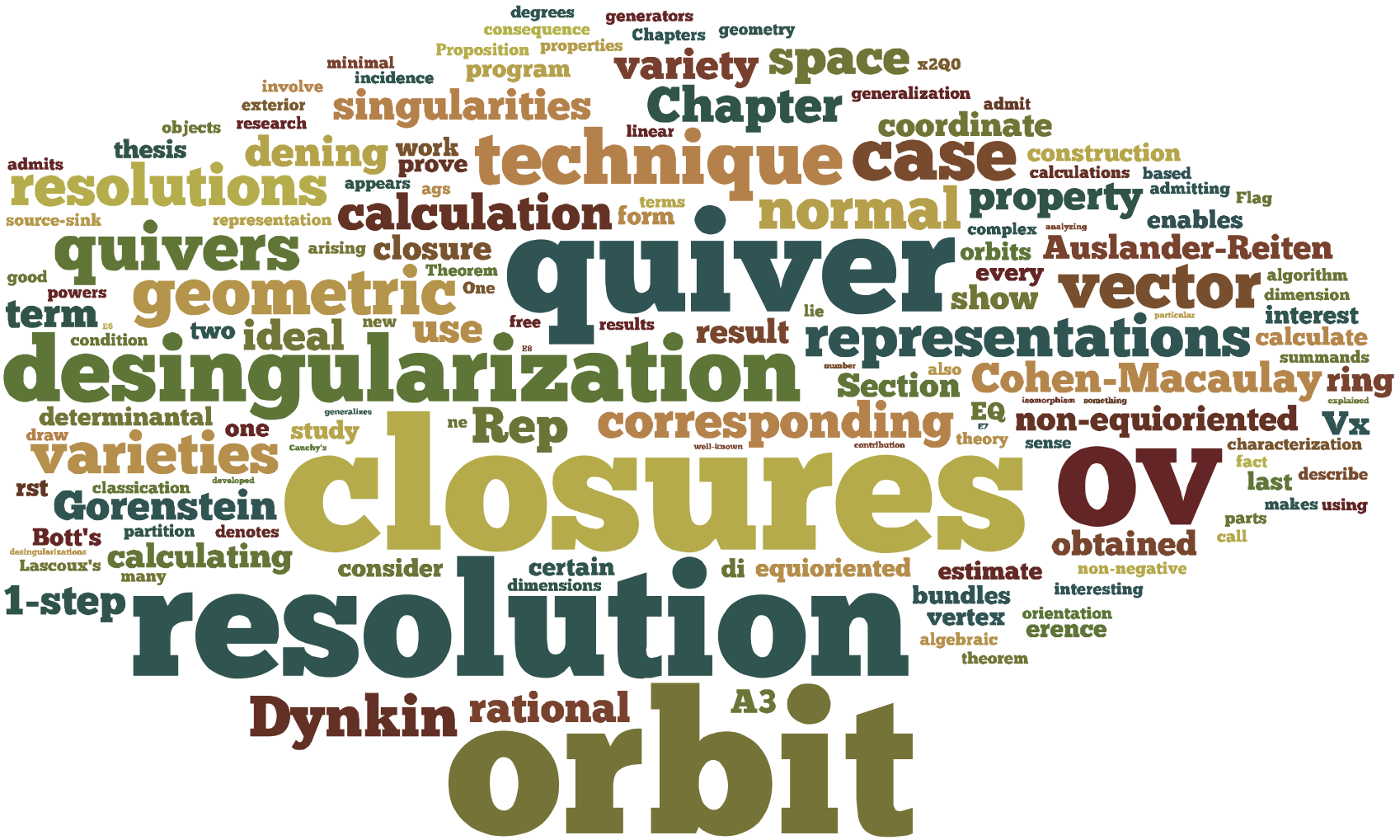
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Orbit closures of quiver representations

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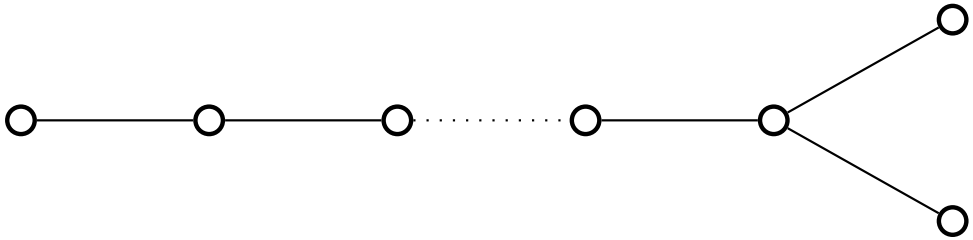
September 5, 2012

Outline

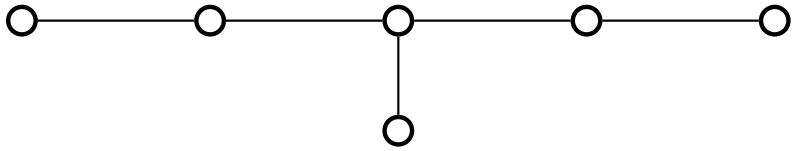
- Quiver $Q = (Q_0, Q_1)$ is a directed graph with set of vertices Q_0 and set of arrows Q_1. (Notation: $ta \xrightarrow{a} ha$)
- Dynkin quiver - underlying unoriented graph is a Dynkin diagram.

A_n

D_n

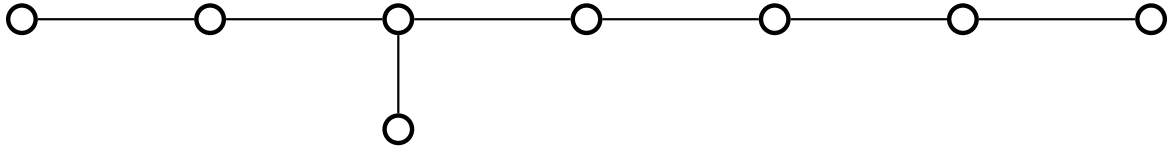


E_6



E_7

E_8



- Quiver $Q = (Q_0, Q_1)$ is a directed graph with set of vertices Q_0 and set of arrows Q_1. (Notation: $ta \xrightarrow{a} ha$)
- Dynkin quiver - underlying unoriented graph is a Dynkin diagram.
- A representation of Q is a pair $V = ((V_i)_{i \in Q_0}, (V(a))_{a \in Q_1})$, where V_i are finite-dimensional vector spaces over algebraically closed field k and $V_{ta} \xrightarrow{V(a)} V_{ha}$.
- Dimension vector $\underline{d} = (\dim V_i)_{i \in Q_0}$.
- The representation space of dimension type $\underline{d} = (d_i)_{i \in Q_0} \in \mathbb{N}^{Q_0}$

$$\begin{aligned} \mathrm{Rep}(Q, \underline{d}) &= \{V = ((V_i)_{i \in Q_0}, (V(a))_{a \in Q_1}) | \dim(V_i) = d_i\} \\ &= \prod_{a \in Q_1} \mathrm{Hom}(V_{ta}, V_{ha}) \cong \mathbb{A}^N. \end{aligned}$$

- $\mathrm{GL}(\underline{d}) = \prod_{i \in Q_0} \mathrm{GL}(d_i)$
- $\mathrm{GL}(\underline{d})$ acts on $\mathrm{Rep}(Q, \underline{d})$ by simultaneous change of basis at each vertex

$$((g_i)_{i \in Q_0}, (V(a))_{a \in Q_1}) = (g_{ha}\, V(a)\, g_{ta}^{-1})_{a \in Q_1}$$

$$\{\mathrm{GL}(\underline{d}) - \text{orbits}\} \longleftrightarrow \{\text{isomorphism classes of representations of } Q\}$$

$$O_V \longleftrightarrow [V]$$

Orbit closure $\overline{O}_V$ is a subvariety of the affine space $\mathrm{Rep}(Q, \underline{d})$.

Example

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

$$Q = (Q_0, Q_1); Q_0 = \{1, 2, 3\}, Q_1 = \{a, b\}$$

$$V_1 \xrightarrow{V_a} V_2 \xrightarrow{V_b} V_3$$

$$Rep(Q, \underline{d}) = \mathrm{Hom}(V_1, V_2) \times \mathrm{Hom}(V_2, V_3)$$

$$\underline{d} = (dim\ V_1, dim\ V_2, dim\ V_3)$$

$$k^3 \xrightarrow{V_a} k^4 \xrightarrow{V_b} k^3$$

$$Rep(Q, (3, 4, 3)) = \mathrm{Hom}(k^3, k^4) \times \mathrm{Hom}(k^4, k^3) \cong \mathbb{A}^{24}$$

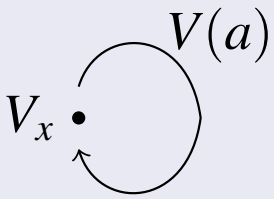
$$\begin{tikzcd} k^3 \arrow[loop below, "Gl(3,k)"] \arrow[r, "V_a"] & k^4 \arrow[loop below, "Gl(4,k)"] \arrow[r, "V_b"] & k^3 \arrow[loop below, "Gl(3,k)"] \end{tikzcd}$$

$$GL(\underline{d}) = GL(3) \times GL(4) \times GL(3)$$

$$(g_1, g_2, g_3) \cdot (V_a, V_b) := (g_2\ V_a\ g_1^{-1},\ \ g_3\ V_b\ g_2^{-1})$$

$\overline{O}_V$ is an affine variety in $\mathbb{A}^{24}$.

Example



$\dim V_x = d$

- $\mathrm{Rep}(Q,d) = \mathbb{M}_{d\times d}(k)$
- Group action: conjugation
- Orbits: conjugacy classes of matrices in $\mathbb{M}(d,k)$
- Geometry: normal, Cohen-Macaulay varieties with rational singularities.
- For nilpotent $V(a)$,
  - if char $k > 0$ then $\overline{O}_{V(a)}$ is a Frobenius split variety.
  - if char $k = 0$ then $\overline{O}_{V(a)}$ is Gorenstein; defining ideal generated by minors of various sizes.

Example

$$\bullet_{V_1} \xrightarrow{V(a)} \bullet_{V_2}$$

$\underline{d} = (d_1, d_2)$

- $\mathrm{Rep}(Q, d) = \mathbb{M}_{d_2 \times d_1}(k)$
- Group action: $(g_1, g_2) * V(a) = g_1 V(a) g_2^{-1}$
- Orbits: O_r = matrices of rank r, $0 \leq r \leq m = \min(d_1, d_2)$
- $\overline{O}_r = \bigcup_{j \leq r} O_j$ (determinantal varieties)

  $= \{W \in \mathrm{Rep}(Q, d) \mid \mathrm{rank}\ W(a) \leq r\}$
- Geometry: normal, Cohen-Macaulay varieties with rational singularities; Gorenstein if $r = 0$, $r = m$ or $d_1 = d_2$; regular if $r = 0$ or $r = m$;
- Resolution of defining ideal (Lascoux); defining ideal generated by $(r+1) \times (r+1)$ minors.

Why study orbit closures?

- types of singularities
- degenerations
- desingularization
- normality, C-M, unibranchness
- tangent spaces
- defining equations
- etc.

Some results

(1981) Abeasis, Del Fra, Kraft

Equi-oriented quiver of type A_n: orbit closures are normal, Cohen-Macaulay with rational singularities.

(1998) Laxmibai, Magyar

Extended the above result to arbitrary characteristic.

(2001,2002) Bobinski, Zwara

Proved the above result for A_n with arbitrary orientation and for D_n.

- Defining ideals?
- What about quivers of type E?

Goal

To study orbit closures by calculating resolutions

Strategy

Use geometric technique

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, "q"] & X \times \mathcal{V} \arrow[r, "p"] \arrow[d, "q"] & \mathcal{V} \\ Y \arrow[r, hook] & X & \end{tikzcd}$$

- X: affine space
- Y: subvariety
- $\mathcal{V}$: a projective variety

$$\begin{tikzcd}
Z \arrow[d, "q'"'] & \subset & X \times \mathcal{V} \arrow[d, "q"] \arrow[dr, "p"] & & \\
Y & \subset & X & & \mathcal{V} \quad \text{(projective variety)} \\
 & & \text{(affine space)} & &
\end{tikzcd}$$

- Let $Z = \mathrm{tot}(\eta)$ and $X \times \mathcal{V} = \mathrm{tot}(\mathcal{E})$.
- Exact sequence of vector bundles over $\mathcal{V}$:

$$0 \to \eta \to \mathcal{E} \to \tau \to 0$$

- Define $\xi = \tau^*$

$$K(\xi)_\bullet : 0 \to \bigwedge^t \xi \to \cdots \to \bigwedge^2 (p^*\xi) \to p^*\xi \to \mathcal{O}_{X \times \mathcal{V}}$$

resolves $\mathcal{O}_Z$ as $\mathcal{O}_{X \times \mathcal{V}}$-module.

$$\begin{tikzcd}
Z \arrow[d, "q'"'] & \subset X \times \mathcal{V} \arrow[d, "q"'] \arrow[dr, "p"] & \\
Y & \subset X & \mathcal{V} \quad \text{(projective variety)} \\
 & \text{(affine space)} &
\end{tikzcd}$$

- Let $Z = \mathrm{tot}(\eta)$ and $X \times \mathcal{V} = \mathrm{tot}(\mathcal{E})$.
- Exact sequence of vector bundles over $\mathcal{V}$:

$$0 \to \eta \to \mathcal{E} \to \tau \to 0$$

- Define $\xi = \tau^*$

$$K(\xi)_\bullet : 0 \to \bigwedge^t \xi \to \cdots \to \bigwedge^2 (p^*\xi) \to p^*\xi \to \mathcal{O}_{X \times \mathcal{V}}$$

$$\downarrow q_*$$

$$\mathbf{F}_\bullet$$

$$\begin{tikzcd} Z \arrow[d, "q'"'] & \subset & X \times \mathcal{V} \arrow[d, "q"] \arrow[dr, "p"] & \\ Y & \subset & X & \mathcal{V} \end{tikzcd}$$

(projective variety)

(affine space)

Main theorem [Weyman]

- $F_i = \bigoplus_{j \geq 0} H^j(\mathcal{V}, \bigwedge^{i+j} \xi) \otimes_K A(-i-j)$ where $A = K[X]$.
- $F_i = 0$ for $i < 0 \Rightarrow \mathbf{F}_\bullet$ is a finite free resolution of the normalization of $K[Y]$.
- If $F_i = 0$ for $i < 0$ and $F_0 = A \Rightarrow Y$ is normal, Cohen-Macaulay and has rational singularities.

(Reineke desingularization)

$$\begin{tikzcd}
Z \subset Rep(Q,\underline{d}) \times \prod_{x\in Q_0} Flag(d_s(x),...,K^{d(x)}) \arrow[d, "q'"'] \arrow[d, shift left=6, "q"] \arrow[dr, "p"] & \\
\overline{O}_V \subset Rep(Q,\underline{d}) & \prod_{x\in Q_0} Flag(d_s(x),...,K^{d(x)})
\end{tikzcd}$$

- $F_i = \bigoplus_{j\geq 0} H^j(\mathcal{V}, \bigwedge^{i+j}\xi) \otimes_K A(-i-j)$ where $A = K[\mathrm{Rep}(Q,\underline{d})]$.
- If $F_i = 0$ for $i < 0 \Rightarrow \mathbf{F}_\bullet$ is a finite free resolution of the normalization of $K[\overline{O}_V]$.
- If $F_i = 0$ for $i < 0$ and $F_0 = A \Rightarrow \overline{O}_V$ is normal and has rational singularities.

Steps in the construction

- Construct the Auslander-Reiten quiver of Q.
- Find a directed partition of the AR quiver (with s parts).
- Define a subset of $Z_{\mathcal{I}_*,V}$ of $\mathrm{Rep}(Q,\underline{d}) \times \prod_{x\in Q_0} \mathrm{Flag}(\underline{\beta}, V_x)$

$$Z_{\mathcal{I}_*,V} = \{(V, (R_s(x) \subset \cdots \subset R_2(x) \subset V_x)_{x\in Q_0}) \\ \mid \forall a \in Q_1, \forall t,\ \ V(a)(R_t(ta)) \subset R_t(ha)\}$$

$$\begin{tikzcd} Z_{\mathcal{I}_*,V} \arrow[r] \arrow[d, "q"] & \mathrm{Rep}(Q,\underline{d}) \times \prod_{x\in Q_0} \mathrm{Flag}(\underline{\beta}, V_x) \arrow[d, "q"] \\ \overline{O}_V \arrow[r] & \mathrm{Rep}(Q,\underline{d}) \end{tikzcd}$$

Theorem [Reineke]

- $q(Z_{\mathcal{I}_*,V}) = \overline{O}_V$.
- q is a proper birational isomorphism of $Z_{\mathcal{I}_*,V}$ and $\overline{O}_V$.

Example - determinantal variety

$$1 \longrightarrow 2$$

$$\begin{tikzcd} & KK \arrow[dr] & \\ 0K \arrow[ur] & & K0 \end{tikzcd}$$

$$V = a(0K) \oplus b(KK) \oplus c(K0)$$

$$\beta_i = \dim R_i$$

$$Z: \begin{array}{ccc} V_1 & \longrightarrow & V_2 \\ \bigcup & & \bigcup \\ R_1 & \longrightarrow & R_2 \end{array}$$

$$Z \in \mathrm{Rep}(Q, d) \times \prod_{i=1}^{2} \mathrm{Flag}(\beta_i, V_i)$$

$$Z \in \mathrm{Hom}(V_1, V_2) \times \mathrm{Gr}(\beta_2, V_2)$$

Example - determinantal variety

$$V_1 \xrightarrow{\phi} V_2$$
$$\dim V_1 = 2$$
$$\dim V_2 = 3$$
$$r = \operatorname{rank} \phi = 2$$

$$X = \operatorname{Hom}(V_1, V_2) = V_1^* \otimes V_2$$
$$\text{Orbit closure } \overline{O}_V = \{\phi : V_1 \to V_2 \mid \operatorname{rk} \phi \leq 2\}$$
$$\mathcal{V} = \operatorname{Gr}(2, V_2)$$
$$Z = \operatorname{tot}(V_1^* \otimes \mathcal{R}) = \{(\phi, R) \mid \operatorname{im}(\phi) \subseteq R\}$$
$$A = K[X] = \operatorname{Sym}(V_1 \otimes V_2^*)$$

Lascoux resolution

$$0 \longrightarrow S_{21}V_1 \otimes \wedge^3 V_2^* \otimes A(-3) \xrightarrow{d_2} \wedge^2 V_1 \otimes \wedge^2 V_2^* \otimes A(-2) \xrightarrow{d_1} A \longrightarrow 0$$

$$d_1 = (x_{11}x_{22} - x_{12}x_{21},\ x_{11}x_{32} - x_{12}x_{31},\ x_{21}x_{32} - x_{22}x_{31})$$

$$d_2 = \begin{pmatrix} x_{31} & -x_{32} \\ -x_{21} & -x_{22} \\ x_{11} & x_{12} \end{pmatrix}$$

Non-equioriented A_3

$$1 \longrightarrow 2 \longleftarrow 3$$

$$\begin{tikzcd}
 & 0KK \arrow[dr] & & K00 \\
0K0 \arrow[ur] \arrow[dr] & & KKK \arrow[ur] \arrow[dr] & \\
 & KK0 \arrow[ur] & & 00K
\end{tikzcd}$$

$$V = \bigoplus_{\alpha \in R^+} m_\alpha X_\alpha$$

$$\beta_i = \dim R_i$$

$$\begin{tikzcd}
Z\colon & V_1 \arrow[r] & V_2 & V_3 \arrow[l] \\
 & \bigcup & \bigcup & \bigcup \\
 & R_1 \arrow[r] & R_2 & R_3 \arrow[l]
\end{tikzcd}$$

$$Z \subset \mathrm{Rep}(Q, \underline{d}) \times \prod_{i=1}^{3} \mathrm{Flag}(\beta_i, V_i)$$

Non-equioriented A_3

$$1 \longrightarrow 2 \longleftarrow 3$$

$$\begin{tikzcd}
 & 0KK \arrow[dr] & & K00 \\
0K0 \arrow[ur] \arrow[dr] & & KKK \arrow[ur] \arrow[dr] & \\
 & KK0 \arrow[ur] & & 00K
\end{tikzcd}$$

$$V = \bigoplus_{\alpha \in R^+} X_\alpha$$

$$\underline{d} = (3,4,3)$$

$$\underline{\beta} = (2,1,2)$$

$$Z\colon \begin{array}{ccccc}
V_1 & \longrightarrow & V_2 & \longleftarrow & V_3 \\
\bigcup & & \bigcup & & \bigcup \\
R_1 & \longrightarrow & R_2 & \longleftarrow & R_3
\end{array}$$

$$\xi = \mathcal{R}_1 \otimes \mathcal{Q}_2^* \oplus \mathcal{R}_3 \otimes \mathcal{Q}_2^*$$

A_4 (source-sink)

$$1 \longleftarrow 2 \longrightarrow 3 \longleftarrow 4$$

$$\begin{tikzcd}
K000 \arrow[dr] & & 0KK0 \arrow[dr] & & 000K \\
& KKK0 \arrow[ur] \arrow[dr] & & 0KKK \arrow[ur] \arrow[dr] & \\
00K0 \arrow[ur] \arrow[dr] & & KKKK \arrow[ur] \arrow[dr] & & 0K00 \arrow[ur] \\
& 00KK \arrow[ur] & & KK00 \arrow[ur] &
\end{tikzcd}$$

$$V = \bigoplus_{\alpha \in R^+} m_\alpha X_\alpha$$

$$Z\colon \begin{array}{ccccccc}
V_1 & \longleftarrow & V_2 & \longrightarrow & V_3 & \longleftarrow & V_4 \\
\bigcup & & \bigcup & & \bigcup & & \bigcup \\
R_1 & \longleftarrow & R_2 & \longrightarrow & R_3 & \longleftarrow & R_4 \\
\bigcup & & \bigcup & & \bigcup & & \bigcup \\
S_1 & \longleftarrow & S_2 & \longrightarrow & S_3 & \longleftarrow & S_4
\end{array}$$

$$Z \subset \mathrm{Rep}(Q, \underline{d}) \times \prod_{i=1}^{4} \mathrm{Flag}(\beta_i^2, \beta_i^1, V_i)$$

A_4 (source-sink)

$$1 \longleftarrow 2 \longrightarrow 3 \longleftarrow 4$$

$$\begin{tikzcd}
K000 \arrow[dr] & & 0KK0 \arrow[dr] & & 000K \\
& KKK0 \arrow[ur] \arrow[dr] & & 0KKK \arrow[ur] \arrow[dr] & \\
00K0 \arrow[ur] \arrow[dr] & & KKKK \arrow[ur] \arrow[dr] & & 0K00 \\
& 00KK \arrow[ur] & & KK00 \arrow[ur] &
\end{tikzcd}$$

$$V = \bigoplus_{\alpha \in R^+} m_\alpha X_\alpha$$

$$Z\colon \begin{array}{ccccccc}
V_1 & \longleftarrow & V_2 & \longrightarrow & V_3 & \longleftarrow & V_4 \\
\cup & & \cup & & \cup & & \cup \\
R_1 & \longleftarrow & R_2 & \longrightarrow & R_3 & \longleftarrow & R_4 \\
\cup & & \cup & & \cup & & \cup \\
S_1 & \longleftarrow & S_2 & \longrightarrow & S_3 & \longleftarrow & S_4
\end{array}$$

$$\xi = (\mathcal{R}_2^1 \otimes \mathcal{Q}_1^{1*} + \mathcal{R}_2^2 \otimes \mathcal{Q}_1^{2*}) \oplus (\mathcal{R}_2^1 \otimes \mathcal{Q}_3^{1*} + \mathcal{R}_2^2 \otimes \mathcal{Q}_3^{2*}) \oplus (\mathcal{R}_4^1 \otimes \mathcal{Q}_3^{1*} + \mathcal{R}_4^2 \otimes \mathcal{Q}_3^{2*})$$

- $\xi = \bigoplus_{a \in Q_1} \left(\sum_{t=1}^{s} \mathcal{R}_t^{ta} \otimes \mathcal{Q}_t^{ha*} \right) \xi = \bigoplus_{a \in Q_1} \left(\mathcal{R}_t^{ta} \otimes \mathcal{Q}_t^{ha*} \right)$
- $\bigwedge^{t} \xi = \bigoplus_{\sum_{a \in Q_1} |\lambda(a)| = t} S_{\lambda(a)} \mathcal{R}_t^{ta} \otimes S_{\lambda(a)'} \mathcal{Q}_t^{ha*}$
- Assign weights to each summand.
- Apply Bott's theorem to these weights to calculate

$$F_i = \bigoplus_{j \geq 0} H^j(\mathcal{V}, \bigwedge^{i+j} \xi) \otimes_K A(-i-j), \quad j = \#\text{exchanges}$$

- $D(\underline{\lambda}) := \sum_{a \in Q_1} |\lambda(a)| - j = (i+j) - j = i.$
- Rewrite

$$F_{D(\underline{\lambda})} = \bigoplus_{j \geq 0} H^j(\mathcal{V}, \bigwedge^{i+j} \xi) \otimes_K A(-i-j)$$

Non-equioriented A_3

$$V_1 \longrightarrow V_3 \longleftarrow V_2$$

$$F_{D(\underline{\lambda})} = \bigoplus_{|\lambda|+|\mu|=t} c^{\nu}_{\lambda,\mu}(S_\lambda V_1 \otimes S_\mu V_2 \otimes S_\nu V_3^*)$$

Theorem (-)

- $D(\underline{\lambda}) \geq E_Q$
- $F_0 = A$ and $F_i = 0$ for $i < 0$. Thus $\overline{O}_V$ is normal and has rational singularities.

$$V_1 \xrightarrow{\phi} V_3 \xleftarrow{\psi} V_2$$

Theorem(-)

- Minimal generators of the defining ideal:

  Let p =rank(ϕ), q =rank(ψ), r =rank($\phi + \psi$), the minimal generators of $\overline{O}_V$ are

  - $(p+1) \times (p+1)$ minors of ϕ,
  - $(q+1) \times (q+1)$ minors of ψ,
  - $(r+1) \times (r+1)$ minors of $\phi + \psi$, taken by choosing $b+k+1$ columns of ϕ and $c+l+1$ columns of ψ such that $k+l = d-1$. (b, c, d are multiplicities of some indecomposable representations)

- Characterization of Gorenstein orbits.

$$1 \longrightarrow 2 \longleftarrow 3$$

$$\begin{tikzcd}
 & 0KK \arrow[dr] & & K00 \\
0K0 \arrow[ur] \arrow[dr] & & KKK \arrow[ur] \arrow[dr] & \\
 & KK0 \arrow[ur] & & 00K
\end{tikzcd}$$

$$V = \bigoplus_{\alpha \in R^+} X_\alpha$$

$$\underline{d} = (3,4,3)$$

$$\begin{tikzcd}
Z\colon & V_1 \arrow[r] & V_2 & V_3 \arrow[l] \\
 & R_1 \arrow[r] \arrow[u, phantom, "\bigcup"] & R_2 \arrow[u, phantom, "\bigcup"] & R_3 \arrow[l] \arrow[u, phantom, "\bigcup"]
\end{tikzcd}$$

$$\xi = \mathcal{R}_1 \otimes \mathcal{Q}_2^* \oplus \mathcal{R}_3 \otimes \mathcal{Q}_2^*$$

$$A = \mathrm{Sym}(V_1 \otimes V_3^*) \otimes \mathrm{Sym}(V_2 \otimes V_3^*)$$

$$A$$

$$\uparrow$$

$$(\wedge^3 V_1 \otimes \wedge^3 V_3^* \otimes A(-3)) \oplus (\wedge^3 V_2 \otimes \wedge^3 V_3^* \otimes A(-3)) \oplus$$
$$(\wedge^2 V_1 \otimes \wedge^2 V_2 \otimes \wedge^4 V_3^* \otimes A(-4))$$

$$\uparrow$$

$$(S_{211} V_1 \otimes \wedge^4 V_3^* \otimes A(-4)) \oplus (S_{211} V_2 \otimes \wedge^4 V_3^* \otimes A(-4)) \oplus$$
$$(\wedge^3 V_1 \otimes \wedge^2 V_2^* \otimes S_{2111} V_3^* \otimes A(-5)) \oplus (\wedge^2 V_1 \otimes \wedge^3 V_2 \otimes S_{2111} V_3^* \otimes A(-5)) \oplus$$
$$\wedge^3 V_1 \otimes \wedge^3 V_2 \otimes S_{222} V_3^* \otimes A(-6)$$

$$\uparrow$$

$$(S_{211} V_1 \otimes \wedge^3 V_2 \otimes S_{2221} V_3^* \otimes A(-7)) \oplus (\wedge^3 V_1 \otimes S_{211} V_2 \otimes S_{2221} V_3^* \otimes A(-7)) \oplus$$
$$(\wedge^2 V_1 \otimes S_{222} V_2^* \otimes S_{2222} V_3^* \otimes A(-8)) \oplus (S_{222} V_1 \otimes \wedge^2 V_2 \otimes S_{2222} V_3^* \otimes A(-8)) \oplus$$
$$\wedge^3 V_1 \otimes \wedge^3 V_2 \otimes S_{3111} V_3^* \otimes A(-6)$$

$$\uparrow$$

$$(S_{211} V_1 \otimes S_{211} V_2 \otimes S_{2222} V_3^* \otimes A(-8)) \oplus (S_{222} V_1 \otimes \wedge^3 V_2 \otimes S_{3222} V_3^* \otimes A(-9)) \oplus$$
$$(\wedge^3 V_1 \otimes S_{222} V_2 \otimes S_{3222} V_3^* \otimes A(-9))$$

$$\uparrow$$

$$(S_{222} V_1 \otimes S_{222} V_2 \otimes S_{3333} V_3^* \otimes A(-12))$$

Source-sink quivers

Theorem(-)

Q be a source-sink quiver and V be a representation of Q such that the orbit closure $\overline{O}_V$ admits a 1-step desingularization Z. Then

$$D(\underline{\lambda}) \geq E_Q$$

Corollary

If Q is Dynkin then $\overline{O}_V$ is normal and has rational singularities.

Orbit closures of type E_6, E_7 and E_8 admitting a 1-step desingularization are normal!

Corollary

If Q is extended Dynkin then $\mathbf{F}_\bullet$ is a minimal free resolution of the normalization of $\overline{O}_V$.

Equioriented A_n

Theorem(-)

Let Q be an equioriented quiver of type A_n and β, γ be two dimension vectors. Let $\alpha = \beta + \gamma$ and $V \in \mathrm{Rep}(Q, \alpha)$. Then

- $D(\underline{\lambda}) \geq E_Q$
- $F(\beta, \gamma)_i = 0$ for $i < 0$.
- $F(\beta, \gamma)_0 = A$.
- The summands of $F(\beta, \gamma)_1$ are of the form $\bigwedge^{\gamma_i + \beta_j + 1} V_i \otimes \bigwedge^{\gamma_i + \beta_j + 1} V_j^*$ where $1 \leq i < j \leq n$.

Thank you!