quasi-coherent sheaves.

Note Title 12/17/2012

S is a graded ring.

$\text{Proj } S = \{$homog. primes of S except for $S_+ = S_1 \oplus S_2 \oplus \cdots\}$

Basis for Zariski topology -
$\{D(f) \mid f$ is homogeneous$\}$

$\mathcal{O}_X(D(f)) := S_{(f)}$ where

$S_{(f)} =$ homog. elements of deg. 0. in S_f.

$\{S_f : \{\frac{a}{f^i}\}$, $\deg\left(\frac{a}{f^i}\right) = \deg a - i \deg f$

$\therefore$ deg. 0 elts. means elts of S_f whose num & den. have same degree.$\}$

Let $X = (\text{Proj } S, \mathcal{O}_X)$.

- An $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ of abelian gps. on X such that $\mathcal{F}(\mathcal{U})$ is an $\mathcal{O}_X(\mathcal{U})$-module satisfying:

$\forall V \subseteq \mathcal{U}$, the maps are compatible

$$\begin{array}{ccc} \mathcal{F}(\mathcal{U}) & \longrightarrow & \mathcal{F}(V) \\ \wr & & \wr \\ \mathcal{O}_X(\mathcal{U})\text{-module} & & \mathcal{O}_X(V)\text{-module} \end{array}$$

• Let M be a graded S-module

Define an $\mathcal{O}_X$-module $\tilde{M}$:

$$\tilde{M}(D(f)) := M_{(f)}$$

Examples

1) $S = k[x_0, x_1]$ $\qquad$ k: a field.

$$\text{Proj } S = \mathbb{P}^1_k = \{(a_0, a_1) \,/\, a_0, a_1 \in k - \{0\}\} \Big/ (a_0; a_1) \sim (\lambda a_0, \lambda a_1)$$

$\forall \lambda \neq 0$ in k.

$U_0 = D(x_0) = \{$primes in Proj S not containing $x_0\}$

$U_1 = D(x_1)$

Then $\quad U_0 \simeq \text{Spec } k\left[\frac{x_1}{x_0}\right] \left(= S_{(x_0)} = \left\{ \left(\frac{x_1}{x_0}\right)^i \right\} \right)$

$\qquad\quad U_1 \simeq \text{Spec } k\left[\frac{x_0}{x_1}\right]$

Let $M = S(1)$, M is a graded S-module.

What is $\widetilde{S(1)}$?

$$\widetilde{S(1)}(D(x_0)) = S(1)_{(x_0)} = \frac{\overset{(\deg i \text{ elt. of } S(1))}{\deg i{+}1 \text{ elt. of } S}}{x_0^i}$$

$$= \left\{ \frac{x_0^{i+1}}{x_0^i}, \frac{x_1^{i+1}}{x_0^i} \right\}$$

$$\widetilde{S(-1)}\left(D(x_0)\right) = S(-1)_{(x_0)} = \left\{ \frac{x_0^{i-1}}{x_0^{i}} = \frac{1}{x_0} \right\}$$

$$\boxed{\mathcal{O}_{\mathbb{P}^1}(n) := \widetilde{S(n)}}$$

$X = \operatorname{Proj} k[X_0, X_1]$. **Krishna - 2**

Examples of $\mathcal{O}_X$-modules on X -

1) $M = S(1)$ is a graded module over S.

Consider $\tilde{M} = \mathcal{O}_X(1)$.

$\tilde{M}(\mathcal{U}_0) = S(1)_{(X_0)}$, $\tilde{M}(\mathcal{U}_1) = S(1)_{(X_1)}$

(clear from the definition)

Let's calculate $\tilde{M}(X)$ i.e. global sections of $\tilde{M}$.

If we can find sections over $\mathcal{U}_0$ & $\mathcal{U}_1$ which agree on $\mathcal{U}_0 \cap \mathcal{U}_1$, then those are the sections we want.

$$\left\{ \begin{array}{l} \text{recall} \\ S(1)_{(X_0)} = \left\{ \frac{f}{x_0^n} \;\middle|\; f \text{ is homog. of deg. } n+1 \right\} \end{array} \right\}$$

$\mathcal{U}_0 \cap \mathcal{U}_1 = D(X_0 X_1) = \{\text{primes which do not contain } X_0 X_1\}$

$\therefore \tilde{M}(\mathcal{U}_0 \cap \mathcal{U}_1) = S(1)_{(X_0 X_1)}$

We have maps

- $\tilde{M}(\mathcal{U}_0) \longrightarrow \tilde{M}(\mathcal{U}_0 \cap \mathcal{U}_1)$

$$S(1)_{(x_0)} \longrightarrow S(1)_{(x_0 x_1)}$$

$$\frac{f}{x_0^n} \longmapsto \frac{f x_1^n}{x_0^n x_1^n} \qquad \deg f = n+1$$

- $S(1)_{(x_1)} \longrightarrow S(1)_{(x_0 x_1)}$

$$\frac{g}{x_1^m} \longmapsto \frac{g x_0^m}{x_0^m x_1^m} \qquad \deg g = m+1.$$

If $\dfrac{f x_1^n}{x_0^n x_1^n} = \dfrac{g x_0^m}{x_0^m x_1^m}$ in $S(1)_{(x_0 x_1)}$

then $x_0^m x_1^{n+m} f = x_0^{n+m} x_1^n g$

$$\Rightarrow f_1 x_1^m = g x_0^n$$

$$\therefore \left.\begin{array}{ll} \text{(i) } f_1 = x_0^{n+1} & g = x_1^m x_0 \\ \text{(ii) } f_1 = x_1 x_0^n & g_2 = x_1^{m+1} \end{array}\right\}$$

These choices give sections $\left\{\begin{array}{l} \frac{x_0}{1} \text{ on } \mathcal{U}_0 \text{ \& } \frac{x_0}{1} \text{ on } \mathcal{U}_1 \\ \frac{x_1}{1} \text{ on } \mathcal{U}_0 \text{ \& } \frac{x_1}{1} \text{ on } \mathcal{U}_1 \end{array}\right.$

$\therefore \tilde{M}(X) =$ v.s. over k generated by x_0, x_1.

Fact: global sections form a f.d. v.s. over the underlying field.

2) $\widetilde{M} = \widetilde{S(-1)}$

Repeat same steps with $\deg f_1 = n-1$,

$\deg g = m-1$.

Then $f x_1^m = g x_0^n$ is not satisfied by any choice of f & g.

$\therefore \nexists$ non-zero global section of $\mathcal{O}_X(-1)$.

In general, $\mathcal{O}_X(n)$, $n<0$ has no global sections.

$\mathcal{O}_X(2)$ is generated by 3 sections.

(3-dim'l k-v.s.) $\left\{ \frac{x_0^2}{1}, \frac{x_0 x_1}{1}, \frac{x_1^2}{1} \right\}$.

$\mathcal{O}_X(n)_{n>0}$ is generated by deg. n homogeneous polynomials in x_0, x_1.

General : $\mathcal{O}_{\mathbb{P}^r_k}(n)_{n>0}$ is generated by deg. n homog. poly. in $x_0, x_1, \ldots, x_r$ ($r+1$ vars.)

Sheaf of differentials on $\mathbb{P}^1$:

$$y_0 = \frac{x_1}{x_0} \; ; \; y_1 = \frac{x_0}{x_1} = \frac{1}{y_0}$$

$$\Omega_{\mathbb{P}^1}\big|_{\mathcal{U}_0} = dy_0 \, \mathcal{O}_{\mathcal{U}_0}$$

$$\Omega_{\mathbb{P}^1}\big|_{\mathcal{U}_1} = dy_1 \cdot \mathcal{O}_{\mathcal{U}_1}$$

Want to find dy_0 on $\mathcal{U}_1$:

$$dy_0 = d\left(\frac{1}{y_1}\right) = -\frac{1}{y_1^2} dy_1$$

$$\therefore \underset{\omega_{\mathbb{P}^1}}{\underset{\shortparallel}{\Omega_{\mathbb{P}^1}}} \simeq \mathcal{O}_{\mathbb{P}^1}(-2).$$

Cohomology

$(X, \mathcal{O}_X)$, $\mathcal{F}$: $\mathcal{O}_X$-module, $\Gamma(X,\mathcal{F}) = \mathcal{F}(X)$ global section functor.

Category of $\mathcal{O}_X$-modules $\xrightarrow{\Gamma}$ Category of abelian groups.

$$\mathcal{F} \longmapsto \Gamma(X,\mathcal{F})$$

$\Gamma(X, -)$ is left exact:

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \quad \text{exact}$$

$$\Rightarrow \quad 0 \to \Gamma(X,\mathcal{F}_1) \to \Gamma(X,\mathcal{F}_2) \to \Gamma(X,\mathcal{F}_3)$$

is exact.

Consider the right derived functors -

Let $\mathcal{I}_\bullet : 0 \to \mathcal{F} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \cdots$ (exact) be an injective resolution of $\mathcal{F}$,

then $\Gamma(\mathcal{I}_\bullet) : 0 \to \Gamma(X,\mathcal{F}) \to \Gamma(X,\mathcal{I}_0) \to \cdots$ is a complex.

$$H^i(X,\mathcal{F}) = H^i(\Gamma(\mathcal{I}_\bullet))$$

$$H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F})$$

In general, it is not easy to write an injective resolution. So other techniques are needed for calculating the cohomology groups.

One of such techniques is -

Čech cohomology - We work with an example

$X = \mathbb{P}^r_k$, consider an open cover -
$= \mathrm{Proj}\, S$
$S = k[x_0, \ldots, x_r]$ $\quad \mathcal{U}_i = D(x_i)$

The Čech complex is defined w.r.t. an open cover. The terms of the complex are -

$$C^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(\mathcal{U}_0) \times \mathcal{F}(\mathcal{U}_1) \times \cdots \times \mathcal{F}(\mathcal{U}_r)$$

$$C^1(\mathcal{U}, \mathcal{F}) = \prod_{0 \le i_0 < i_1 \le r} \mathcal{F}(\underbrace{\mathcal{U}_{i_0} \cap \mathcal{U}_{i_1}}_{\text{denote by } \mathcal{U}_{i_0 i_1}})$$

$$C^2(\mathcal{U}, \mathcal{F}) = \prod_{0 \le i_0 < i_1 < i_2 \le r} \mathcal{F}(\mathcal{U}_{i_0 i_1 i_2})$$

$$\vdots$$

$$C^r(\mathcal{U}, \mathcal{F}) = \mathcal{F}(\mathcal{U}_{i_0 i_1 \ldots i_r})$$

$$C^0(\mathcal{F}) \to C^1(\mathcal{F}) \to \cdots \to C^r(\mathcal{F})$$

$$(s_0, s_1, \ldots, s_r) \mapsto \left((s_{i_0} - s_{i_1})_{i_0 < i_1} \right)$$

In general $C^k(\mathcal{F}) \longrightarrow C^{k+1}(\mathcal{F})$

$$(s_{i_0 \ldots i_k}) \longmapsto \left(s_{i_1 i_2 \ldots i_k'}, \; s_{i_0 i_2 \ldots i_k}, \; \ldots \; s_{i_0 \ldots i_{k-1}} \right)$$

(check that this is a complex.)

each of these is an alternating sum of sections on $\mathcal{U}_0 \cap \mathcal{U}_1 \cap \cdots \cap \mathcal{U}_k$, taken k at a time.

* Cohomology of the Čech complex is the sheaf cohomology $H^i(X, \mathcal{F})$.

<u>Example</u> : $X = \mathbb{P}^1$, $\mathcal{U} = \{U_0, U_1\}$, $\mathcal{F} = \mathcal{O}_X(1)$.

$$\mathcal{F}(U_0) = S(1)_{(x_0)}$$

$$\mathcal{F}(U_1) = S(1)_{(x_1)}$$

$$\mathcal{F}(U_0 \cap U_1) = S(1)_{(x_0 x_1)}$$

Čech complex : $\mathcal{F}(U_0) \times \mathcal{F}(U_1) \longrightarrow \mathcal{F}(U_0 U_1)$

$$S(1)_{(x_0)} \times S(1)_{(x_1)} \xrightarrow{d.} S(1)_{(x_0 x_1)}$$

$$(s_0, s_1) \longmapsto \left(\frac{s_0 x_1}{x_1} - \frac{s_1 x_0}{x_0} \right)$$

$H^0(X, \mathcal{F}) = \ker d$, $\quad H^1(X, \mathcal{F}) = \operatorname{coker} d.$

(done yesterday) $\quad = 0$ (check)

<u>Theorem</u> : $X = \mathbb{P}^r_k$, $\mathcal{O}_X(m)$, $m \in \mathbb{Z}$

(1) $H^0(X, \mathcal{O}_X(m)) = \begin{cases} \text{homog. poly. of degree } m \text{ in } S & \text{if } m \geq 0 \\ 0 & \text{if } m < 0 \end{cases}$

(2) $H^i(X, \mathcal{O}_X(m)) = 0 \qquad i \neq 0, r.$

(3) $H^r(X, \mathcal{O}_X(m)) = \begin{cases} H^0(X, \mathcal{O}_X(-r-1-m))^* & m \leq -r-1. \\ 0 & m \geq -r \end{cases}$

<u>Proof</u> : Let $\mathcal{F} = \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_X(m)$

Note that

$\mathcal{F}(U_0) = S_{x_0}$: $\quad \mathcal{O}_X(1)(U_0) = \{\deg 1 \text{ elts in } S_{x_0}\}$

$\qquad \mathcal{O}_X(-1)(U_0) = \{\deg -1 \text{ elts in } S_{x_0}\}$

and so on. $\therefore \bigoplus \mathcal{O}_X(m)(U_0) = S_{x_0}$

Similarly, $\mathcal{F}(U_0 U_1) = S_{x_1 x_j}$

$\vdots$

$\mathcal{F}(U_0 U_1 \cdots U_r) = S_{x_0 \cdots x_r}$

$C^{r-1}(\mathcal{F}) \longrightarrow C^r(\mathcal{F})$ is

$$S_{x_0 x_2 \cdots x_r} \times S_{x_0 x_1 x_3 \cdots x_r} \times \cdots \times S_{x_0 \cdots x_{r-1}} \xrightarrow{d} S_{x_0 \cdots x_r}$$

$$(s_0, s_1, \ldots, s_r) \longmapsto \left(\frac{s_0 x_1}{x_1} - \frac{s_1 x_2}{x_2} + \cdots + \frac{s_r x_0}{x_0}\right)$$

Want a k-basis for $S_{x_0 x_1 \cdots x_r}$

(a term in this ring is $(x_0 x_1 \cdots x_r)^{-l} f(x_0, \ldots, x_r)$)

$\therefore$ a k-basis is -

$$\{ x_0^{l_0} \cdots x_r^{l_r} \mid l_i \in \mathbb{Z} \}.$$

k-basis for Im d is -

$$\left\{ x_0^{l_0} \cdots x_r^{l_r} \;\middle|\; \begin{array}{l} l_i \in \mathbb{Z} \\ l_i \geq 0 \text{ for some } i \end{array} \right\}$$

$\therefore$ coker d has basis

$$= \{ x_0^{l_0} \cdots x_r^{l_r} \mid l_i < 0 \ \forall i \}$$

This is the basis for $H^r(X, \mathcal{F})$.

$$H^r(X, \mathcal{F}) = H^r(X, \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_X(m))$$

$$= \bigoplus_{m \in \mathbb{Z}} H^r(X, \mathcal{O}_X(m))$$

$\therefore$ k-basis of $H^r(X, \mathcal{O}_X(m))$ is $\left\{ x_0^{l_0} \cdots x_r^{l_r} \;\middle|\; \begin{array}{l} l_i < 0 \\ \sum l_i = m \end{array} \right\}$

Note that the max. possible degree is $-r-1$. (the degree of $x_0^{-1} \cdots x_r^{-1}$).

$\therefore H^r(X, \mathcal{O}_X(m)) \neq 0$ for $m \leq -r-1$.

This also proves that the dimensions of vector spaces $H^r(X, \mathcal{O}_X(m))$ and $H^0(X, \mathcal{O}_X(m))$ $(m \leq -r-1)$ are equal.

Let $\mathcal{F} = \tilde{M}$.

$C^i(\mathcal{U}, \mathcal{F}) = (i+1)$-term of $\varinjlim_t K_\bullet(x_0^t, x_1^t, \dots x_r^t; M)$

$$\therefore H^i(X, \mathcal{F}) = \varinjlim_t \underbrace{H^{i+1}(x_0^t, \dots, x_r^t; M)}_{\text{Koszul cohomology}}$$

This can be used to prove that
$$H^i(X, \mathcal{O}_X(m)) = 0 \quad \text{for } i \neq 0, r$$
since the corresponding Koszul homologies are zero.

Krishna - Tutorial 1

1). X : topological space

$\mathcal{B}$: basis of X.

$\mathcal{F}(\mathcal{U})$ is defined $\forall \mathcal{U} \in \mathcal{B}$ such that all the properties of a sheaf are satisfied

Let $\mathcal{U} \subset X$ be open. Define
$$\tilde{\mathcal{F}}(\mathcal{U}) := \varprojlim_{\substack{V \subseteq \mathcal{U} \\ V \in \mathcal{B}}} \mathcal{F}(V)$$

Show that $\tilde{\mathcal{F}}$ is a sheaf and $\tilde{\mathcal{F}}(\mathcal{U}) = \mathcal{F}(\mathcal{U})$ $\forall \mathcal{U} \in \mathcal{B}$

$$\left\{ \begin{array}{l} (s_v)_{v \in \mathcal{U}} \\ \exists \ \varprojlim \mathcal{F}(v) \ {}_{v \in B} \end{array} \quad \text{is such that } \ \rho^{vw}(s_v) = s_w \right\}$$

2) $X = \mathbb{P}^2_k \ , \ \mathcal{O}_X(1) = \mathcal{F}$

Verify by direct computation that

$$H^1(X, \mathcal{O}_X(1)) = H^2(X, \mathcal{O}_X(1)) = 0 .$$

Recall the following:

whenever we have

$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$ be a s.e.s. of $\mathcal{O}_X$-modules

we get l.e.s. of cohomology -

$$0 \to H^0(X, \mathcal{F}_1) \to H^0(X, \mathcal{F}_2) \to H^0(X, \mathcal{F}_3) \to \cdots$$

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3) $\mathbb{P}^2_k \supseteq X = \{p\}$, let $p = (1,0,0)$

$$I_X = (X_1, X_2) \subseteq K[X_0, X_1, X_2] = S$$

$$0 \longrightarrow I_X \longrightarrow S \longrightarrow S/I_X \longrightarrow 0$$

gives

$$0 \longrightarrow \tilde{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

$\otimes \ \mathcal{O}_{\mathbb{P}^2}(m)$ gives -

$$(*) \quad 0 \longrightarrow \mathcal{I}_X(m) \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_X(m) \longrightarrow 0$$

$\mathcal{O}_{\mathbb{P}^2}(m)$ is a locally free $\mathcal{O}_X$-module, $\therefore$ it is flat.

$$\left\{ \mathcal{O}_{\mathbb{P}^2}(m)(D(x_0)) = S(m)_{(x_0)} \simeq S_{(x_0)} \right\}$$

$\therefore$ $(*)$ is a short exact sequence $\forall m$.

Exercise: Find $\dim_k H^i(\mathcal{I}_X(m))$ $\forall i = 0,1,2$
$\forall m \in \mathbb{Z}$

$\boxed{i=0}$

eg: if $m=0$, we have the l.e.s.

$$0 \to H^0(\mathcal{I}_X) \xrightarrow{\varphi} H^0(\mathcal{O}_{\mathbb{P}^2}) \to \cdots$$

$$\therefore H^0(\mathcal{I}_X) \subset \underset{\substack{\| \\ k}}{H^0(\mathcal{O}_{\mathbb{P}^2})} \quad (\text{since } \varphi \text{ is injective})$$

$$\therefore H^0(\mathcal{I}_X) = 0$$

$$H^0(\mathcal{I}_X) = 0$$

Similarly $H^0(\mathcal{I}_X(m)) = 0$ for $m < 0$

$$\therefore \underset{(\dim H^0)}{h^0}(\mathcal{I}_X(m)) = 0$$

For $m=1$:

$$0 \to H^0(\mathcal{I}_X(1)) \to H^0(\mathcal{O}_{\mathbb{P}^2}(1)) = \langle x_0, x_1, x_2 \rangle$$

the deg. 1 terms which vanish on X are x_1 and x_2.

$$\therefore h^0(\mathcal{I}_X(1)) = 2.$$

If $m=2$:

$$0 \to H^0(\mathcal{I}_X(2)) \to H^0(\mathcal{O}_{\mathbb{P}^2}(2)) = \langle x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2 \rangle$$

The terms that vanish on X are: x_1^2, x_2^2, x_0x_1, x_0x_2 and x_1x_2.

$$\therefore h^0(\mathcal{I}_X(2)) = 5.$$

In general, $h^0(\mathcal{I}_X(m)) = h^0(\mathcal{O}_{\mathbb{P}^2}(m)) - 1$
$= \binom{m+1}{2} - 1$. for $m > 0$

$\boxed{i=1}$:

$$0 \to \overset{\substack{0 \\ \|}}{H^0(\mathcal{I}_X)} \to \overset{1}{H^0(\mathcal{O}_{\mathbb{P}^2})} \to \overset{1}{H^0(\mathcal{O}_X)} \to H^1(\mathcal{I}_X) \to \overset{\substack{0 \\ \|}}{H^1(\mathcal{O}_{\mathbb{P}^2})}$$

$$\to H^1(\mathcal{O}_X) \to \cdots$$

$$H^1(\mathcal{O}_{\mathbb{P}^2}(m)) = 0$$

For $m=0$:

$$0 \to H^0(\mathcal{I}_X) = 0 \to H^0(\mathcal{O}_{\mathbb{P}^2}) \to H^0(\mathcal{O}_X) \to H^1(\mathcal{I}_X) \to 0$$

$$\therefore h^1(\mathcal{I}_X) = h^0(\mathcal{O}_X) - h^0(\mathcal{O}_{\mathbb{P}^2})$$

$$= 1 - 1 = 0 \qquad \boxed{\therefore h^1(\mathcal{I}_X) = 0}$$

For $m<0$:

$$0 \to \underset{0}{\underset{\parallel}{H^0(\mathcal{I}_X(m))}} \to H^0(\mathcal{O}_{\mathbb{P}^2}(m)) \to H^0(\mathcal{O}_X(m)) \to H^1(\mathcal{I}_X(m)) \to \underset{0}{\underset{\parallel}{H^1(\mathcal{O}_{\mathbb{P}^2}(m))}} \to H^1(\mathcal{O}_X(m)) \to \cdots$$

$\therefore$ s.e.s:

$$0 \to H^0(\mathcal{O}_{\mathbb{P}^2}(m)) \to H^0(\mathcal{O}_X(m)) \to H^1(\mathcal{I}_X(m)) \to 0$$

$$h^1(\mathcal{I}_X(m)) = h^0(\mathcal{O}_X(m)) - h^0(\mathcal{O}_{\mathbb{P}^2}(m))$$

$$= 1 - h^0(\mathcal{O}_{\mathbb{P}^2}(m))$$

$$= 1 - 0 \qquad (\because h^0(\mathcal{O}_{\mathbb{P}^2}(m)) = 0 \text{ for } m<0)$$

$$= 1.$$

$$\therefore \boxed{h^1(\mathcal{I}_X(m)) = 1 \quad \forall m<0}$$

For $m>0$:

$$0 \to H^0(\mathcal{I}_X(m)) \to H^0(\mathcal{O}_{\mathbb{P}^2}(m)) \to H^0(\mathcal{O}_X(m)) \to H^1(\mathcal{I}_X(m)) \to \underset{0}{\underset{\parallel}{H^1(\mathcal{O}_{\mathbb{P}^2}(m))}} \to \cdots$$

$$-h^0(\mathcal{I}_X(m)) + h^0(\mathcal{O}_{\mathbb{P}^2}(m)) - h^0(\mathcal{O}_X(m)) + h^1(\mathcal{I}_X(m)) = 0$$

$$\therefore h^1(\mathcal{I}_X(m)) = \binom{m+1}{2} - 1 - \binom{m+1}{2} + 1$$

$$= 0$$

$\boxed{i=2}$:   Ans: $h^2(\mathcal{G}_X(m)) = h^2(\Theta_{\mathbb{P}^2}(m))$

$\mathcal{F}$: coherent sheaf on $\mathbb{P}^r_k$

(Recall - $\exists$ an affine open cover $\{\mathcal{U}_i\}$ of $\mathbb{P}^r_k$ such that $\mathcal{F}|_{\mathcal{U}_i} \simeq \widetilde{M}_i$ where M_i is a f.g. A_i-module, $\operatorname{Spec} A_i = \mathcal{U}_i$

$\mathcal{F}(n) := \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X(n)$, $n \in \mathbb{Z}$)

For $i = 0, \ldots, r$, $n \in \mathbb{Z}$ define

$$\gamma_{i,n} := \dim H^i(\mathcal{F}(n)) = h^i(\mathcal{F}(n))$$

$$\begin{array}{ccccccccc|l}
 & \cdots\ \gamma_{r,-r-1} & \gamma_{r,-r} & \gamma_{r,-r+1} & \cdots & & & & & r \\
 & \vdots & \vdots & \vdots & & & & & & \vdots \\
 & & \gamma_{2,-2} & \gamma_{2,-1} & \gamma_{2,0} & \gamma_{2,1} \cdots & & & & \vdots \\
\gamma_{1,-3} & \gamma_{1,-2} & \gamma_{1,-1} & \gamma_{1,0} & \gamma_{1,1} & \gamma_{1,2} \cdots & & & & 1 \\
\gamma_{0,-2} & \gamma_{0,-1} & \gamma_{0,0} & \gamma_{0,1} & \gamma_{0,2} & \gamma_{0,3} \cdots & & & & 0 \\
\hline
\cdots -2 & -1 & 0 & 1 & 2 & 3 \cdots & & & & \uparrow i
\end{array}$$

grade $\xleftarrow{r}$ homological degree

Examples

(1) $X = \mathbb{P}^r_k$, $\mathcal{F} = \mathcal{O}_X$

$$\begin{array}{ccccccc|l}
\cdots \binom{r+3}{3} & \binom{r+2}{2} & r+1 & & & & & r \\
 & & & 1 & r+1 & \binom{r+2}{2} & \binom{r+3}{3} \cdots & 0 \\
\hline
-2 & -1 & & 0 & 1 & 2 & \cdots &
\end{array}$$

If $\gamma_{i,j} = 0$, then the corresponding entry is left blank

(2) $X = \mathbb{P}^2$, $\mathcal{F}$ = Ideal sheaf of a point in $\mathbb{P}^2$

..6	3	1					2
	...1	1	1				1
				2	5	9...	0
	−2	−1	0	1	2		

(3) $\mathbb{P}^3_{\mathbb{Q}}$, X = intersection of two quadrics in $\mathbb{P}^3$.
(curve)
$\mathcal{F}$ = ideal sheaf of X.

Using Macaulay 2 :

									3
10	4	1							
20	16	12	8	4	1				2
									1
					2	8	19	36..	0
			0	1	2	3	4	...	

Theorem (Eisenbud, Shreyer) :
The cohomology table of a coherent sheaf on $\mathbb{P}^n_k$ is a limit of cohomology tables of 'nice' coherent sheaves.

<u>Vector bundle</u> = locally free $\mathcal{O}_X$-module

$\mathcal{F}$ is an $\mathcal{O}_X$-module, $\mathcal{F}$ is said to be locally free if $\exists$ an open cover $\{\mathcal{U}_i\}$ of X such that $\mathcal{F}|_{\mathcal{U}_i}$ is a free $\mathcal{O}_X|_{\mathcal{U}_i}$-module $\forall i$.

$$\text{i.e. } \mathcal{F}|_{\mathcal{U}_i} = \bigoplus_{I_i} \mathcal{O}_X|_{\mathcal{U}_i}$$

If X is connected, then $|I_i|$ is same $\forall i$ and the common number is called the rank of $\mathcal{F}$.

invertible sheaf = locally free sheaf of rank 1.

<u>Prop</u>: If $X = \mathbb{P}^r$ then $\mathcal{O}_X(m)$ is an invertible sheaf $\forall m \in \mathbb{Z}$.

<u>Pf</u>: $\mathcal{O}_X(m)(\mathcal{U}_0) = S(m)_{(x_0)} \simeq S_{(x_0)}$

$$s \mapsto s x_0^m$$

So $\mathcal{O}_X(m)|_{\mathcal{U}_0} \simeq \widetilde{S(m)_{(x_0)}} \simeq \widetilde{S_{(x_0)}} = \mathcal{O}_X|_{\mathcal{U}_0}$

$\square$

Check: if $m \neq n$, $m, n \in \mathbb{Z}_+$, then it is easy to see that $\mathcal{O}_X(m) \not\simeq \mathcal{O}_X(n)$ by looking at global sections.

if $m, n \in \mathbb{Z}_{<0}$, this can be checked by looking at the number of shifts it takes to hit the ring of global sections.

Divisors on $\mathbb{P}^r_k$: $k = \bar{k}$

Function field of $\mathbb{P}^r_k$ is isomorphic to

$F = \left\{ \frac{f}{g} \;\middle|\; \begin{array}{l} f \,\&\, g \text{ are non-zero homog.} \\ \text{poly. of same degree, } g \neq 0 \end{array} \right\} \subseteq K(x_0, \ldots, x_r)$

$f \in S$, homog. of deg. d

S is a UFD, let $f = f_1^{n_1} \cdots f_\ell^{n_\ell}$, irred. decomp.

. Divisors of f :

$(f) = n_1 Y_1 + \cdots + n_\ell Y_\ell$ where

Y_i = zero set of f_i

(irred. subvariety of $\mathbb{P}^r$ of codim. 1.)

. let $s = \frac{f}{g}$, $\underset{\text{principal divisor}}{(s)} := (f) - (g)$

Divisors are formal integers linear combinations of irreducible codim 1 closed subvarieties.

For every polynomial f, we can define divisor (f), and also for every rational func.

S is a UFD , Y_i is prime ideal of ht. 1

$\therefore$ Y_i is a principal ideal.

$\therefore$ $\exists$ a homog. poly. generating Y_i ,

its degree is called degree of Y_i

So now we can define degree of a divisor D :

$\deg D := \sum n_i (\deg Y_i)$

Note that $\deg(f) = \sum n_i (\deg \varphi_i)$

$= \sum n_i (\deg f_i)$

which is the degree of f as a polynomial.

Also $\deg(s) = 0$ $\left(s = \frac{f}{g},\ f, g \text{ homog. of same deg}\right)$

<u>Lemma</u>: $\deg D = 0 \Rightarrow D = (s)$ for some $s \in F$.

Divisor class group of $X = \mathbb{P}^r$

$$\operatorname{Pic}(X) = Cl(X) := \frac{\text{Divisors}}{\text{principal divisors}}$$

$$Cl(X) \xrightarrow{\deg} \mathbb{Z}$$ is bijective
$$D \longmapsto \deg D$$ group homomorphism.

If D is a divisor, D can be described by a data: $(\mathcal{U}_i, f_i)$

If $\mathcal{F}$ is an invertible $\mathcal{O}_X$-module, then $\exists$ data $(\mathcal{U}_i', f_i')$

The data $(\mathcal{U}_i, f_i)$ can be reconciled with the data $(\mathcal{U}_i', f_i')$

(locally free rank 1 sheaf is free)

Using this, one can prove:

$\mathcal{O}_X(m) \longleftrightarrow$ divisor mH

where H = zero set (x_0).

$\therefore \operatorname{Pic} X \simeq \mathbb{Z}$. $\square$

$X = \mathrm{Proj}\, S$

$\mathcal{F}$ be an $\mathcal{O}_X$-module,

$0 \neq s \in \Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F})$

Note that we always have a map -

$$\mathcal{O}_X(U) \longrightarrow \mathcal{F}(U) \qquad \forall\, U \subseteq X \text{ open}$$
$$1 \longmapsto 1 \cdot s|_U$$

So we have a map

$$s : \mathcal{O}_X \longrightarrow \mathcal{F} \quad (\text{an } \mathcal{O}_X\text{-module homomorphism})$$

Thus $\forall$ section $s \in \Gamma(X, \mathcal{F})$ we have a map

$$s : \mathcal{O}_X \longrightarrow \mathcal{F}.$$

$$\mathcal{F}_x = \text{stalk of } \mathcal{F} \text{ at } x$$
$$= \varinjlim_{x \in U} \mathcal{F}(U)$$

Then $\quad s_x : \mathcal{O}_{X,x} \longrightarrow \mathcal{F}_x \quad \forall\, x \in X$

Let $s_0, \dots, s_n \in H^0(X, \mathcal{F})$. Then we have map

$$(s_0, \dots, s_n) : \mathcal{O}_X^{\oplus n+1} \longrightarrow \mathcal{F}$$

__Defn__ : The sections $s_0, \dots, s_n$ __generate $\mathcal{F}$__ if the map $(s_0, \dots, s_n)$ is surjective.

$\iff (s_0)_x, \dots, (s_n)_x$ generate $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module.

Defn : $\mathcal{F}$ is "globally generated" if $\exists$ a finite family of sections $s_i \in \Gamma(X, \mathcal{F})$ which generate $\mathcal{F}$.

Example : $\mathcal{O}_{\mathbb{P}^1}(2) = \mathcal{F}$

We know that $H^0(\mathcal{O}_{\mathbb{P}^1}(2)) = k x_0^2 \oplus k x_0 x_1 \oplus k x_1^2$

(3-dim. v.s.)

question : is $\mathcal{F}$ globally generated?

Yes : $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \xrightarrow{(x_0^2, x_1 x_2, x_1^2)} \mathcal{O}_{\mathbb{P}^1}(2)$ is surjective

(Check over basic open sets $D(x_0), D(x_1)$)

In fact, $\mathcal{O}_{\mathbb{P}^n}(m)$ is globally generated $\forall n \geq 1$.

$\iff m \geq 0$

(If $m < 0$, $H^0(\mathcal{O}_{\mathbb{P}^n}(m)) = 0$, $\therefore$ there are no global sections)

Theorem (Serre) (1) Suppose $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^r_k$. $\exists\, n_0 \geq 0$ such that $\mathcal{F}(n)$ is globally generated $\forall n \geq n_0$.

(2) $\exists\, n_0 \geq 0$ such that $H^i(X, \mathcal{F}(n)) = 0$ $\forall n \geq n_0, i > 0$.

Let $\mathcal{F}$ be a coherent sheaf. Then $\mathcal{F}(n)$ is globally generated for some n. Hence

$$\mathcal{O}_X^{\oplus m} \longrightarrow \mathcal{F}(n) \longrightarrow 0 \quad \text{for some } m$$

Tensor with $\mathcal{O}_X(-n)$:

$$\mathcal{O}_X(-n)^{\oplus m} \longrightarrow \mathcal{F} \longrightarrow 0$$

This is the beginning of a locally free resolution of $\mathcal{F}$. The kernel of this map is also a coherent sheaf. This resolution will end if X is smooth.

<u>Theorem (Grothendieck)</u> Every locally free sheaf E on $\mathbb{P}^1_k$ is a direct sum of invertible sheaves.

$E = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$ for some $a_1, \ldots, a_r \in \mathbb{Z}$

where $r = \text{rank } E$.

<u>Proof</u>: Induction on r.

$r=1$: done.

Let $r \geq 2$: $\exists$ unique $k_0 \in \mathbb{Z}$ such that

$$h^0(E(k_0)) \neq 0 \;\&\; h^0(E(k_0-1)) = 0.$$

{ Can find k such that $E(k)$ is globally generated $\Rightarrow h^0(E(k)) \neq 0$. ($k$ sufficiently large)

For $k' << 0$, $h^0(E(k'))$

$$\begin{aligned} &\overset{\text{Serre duality}}{=} h^r(E^*(-k'-r-1)) \\ &= 0 \end{aligned}$$

$\therefore h^0(E(k')) = 0$ for some k' sufficiently small

Let k_0 be the point where h^0 jumps from 0 to non-zero. }

Let $0 \neq s \in H^0(E(k_0))$

$$0 \to \mathcal{O}_{\mathbb{P}^1} \xrightarrow{s} E(k_0)$$

$$\left\{ \begin{array}{l} s_x : (\mathcal{O}_{\mathbb{P}^1})_x \to (E(k_0))_x, \quad x \in \mathbb{P}^1 \\ \text{Suppose } s_x = 0 \text{ for some } x \\ \text{Then } s \text{ gives a section over} \\ \quad E(k_0) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{I}_x \text{ where } \mathcal{I}_x \text{ is the ideal sheaf of } \{x\} \\ \quad \mathcal{I}_x = \mathcal{O}_{\mathbb{P}^1}(-1) \text{ (check)} \\ \text{But } E(k_0) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) = E(k_0 - 1) \\ \quad \& \ h^0(E(k_0-1)) = 0. \quad \Rightarrow s = 0 \\ \therefore s \text{ is injective} \end{array} \right\}$$

$$0 \to \mathcal{O}_{\mathbb{P}^1} \xrightarrow{s} E(k_0) \to F \to 0 \quad \longrightarrow (*)$$

F is a locally free sheaf of rank $r-1$.

Induction hypothesis $\Rightarrow$

$$F = \mathcal{O}_{\mathbb{P}^1}(b_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(b_{r-1})$$

Need that $(*)$ is a split exact seq.

The obstruction to this splitting is

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1} \otimes F^*)$$

WTS: this is 0.

$$F^* \otimes \mathcal{O}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-b_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-b_{r-1})$$

$$\underbrace{H^1(F^* \otimes \mathcal{O}_{\mathbb{P}^1})}_{\overset{\text{WTS}}{=} 0} = H^1(\mathcal{O}_{\mathbb{P}^1}(-b_1)) \oplus \cdots \oplus H^1(\mathcal{O}_{\mathbb{P}^1}(-b_{r-1}))$$

Consider: $\circledast \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$:

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to E(k_0 - 1) \to F(-1) \to 0$$

Apply $\Gamma(\mathbb{P}^1, -)$:

$$\cdots \longrightarrow \underbrace{H^0(E(k_0-1))}_{=0} \to H^0(F(-1)) \to \overbrace{H^1(\mathcal{O}_{\mathbb{P}^1}(-1))}^{=0} \longrightarrow \cdots$$

$$\Rightarrow H^0(F(-1)) = 0.$$

$$\Rightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(b_i - 1)) = 0 \quad \forall i.$$

$$\Rightarrow b_i - 1 < 0 \Rightarrow b_i < 1$$

$$\therefore H^1(\mathcal{O}_{\mathbb{P}^1}(-b_i)) = 0 \quad \forall i.$$

$\therefore$ $\circledast$ splits and we're done. $\square$

For $r \geq 2$, this thm. is not true, i.e. not all vector bundles on $\mathbb{P}^r$ $(r \geq 2)$ split as direct sum of line bundles (eg. tangent bundle).

Even so, it is difficult to construct small rank, non-split vector bundles on $\mathbb{P}^r_k$ $(r \geq 2)$.

<u>Defn</u>: $\mathcal{F}$ is d-regular $(d \in \mathbb{Z})$ if

$$H^i(\mathcal{F}(d-i)) = 0 \quad \forall i \geq 1.$$

$$\text{reg } \mathcal{F} = \min \{ d \,/\, \mathcal{F} \text{ is } d\text{-regular} \}.$$

$\shortparallel$

smallest column index d such that only non-zero entries in i^{th} column $(i \geq d)$ (if at all) are h^0's (in the cohomology table)

<u>Krishna - 6</u>

$f : X \to Y$ be a map of schemes

$\mathcal{F}$ be an $\mathcal{O}_X$-module, $\mathcal{G}$ be an $\mathcal{O}_Y$-module.

<u>Pushforward</u>: $f_* \mathcal{F}$,

$\mathcal{U} \subseteq Y$ be open; $(f_* \mathcal{F})(\mathcal{U}) := \mathcal{F}(f^{-1}(\mathcal{U}))$.

(note that $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is given as a part of f)

<u>Pull back</u>: $f^* \mathcal{G}$.

$\mathcal{U} \subseteq X$ be open; $\mathcal{U} \longmapsto \varinjlim_{V \supseteq f(\mathcal{U})} \mathcal{G}(V)$

This is a presheaf.

[A presheaf has all the data reqd. for a sheaf but not the gluing.]

The sheaf associated to this presheaf is $f^{-1}\mathcal{G}$.

$f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-module.

($\mathcal{O}_X$ is also an $f^{-1}\mathcal{O}_Y$-module)

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$, this is an $\mathcal{O}_X$-module

In practice, it is useful to look at the <u>local case</u> : $f : \overset{X}{\overset{\parallel}{\operatorname{Spec} B}} \longrightarrow \overset{Y}{\overset{\parallel}{\operatorname{Spec} A}}$.

note that this is true in general, its not a special case.

M be an A-module $\quad \tilde{M}$ is an $\mathcal{O}_Y$-module

N be a B-module $\quad \tilde{N}$ is an $\mathcal{O}_X$-module

$\exists$ map $A \to B$, $\therefore$ N is also an A-module

(i) $f_*(\tilde{N}) \cong \widetilde{N}_A$

(ii) $f^*(\tilde{M}) \cong \widetilde{(M \otimes_A B)}$

<u>Prop</u> : $f : X \to Y$.

(i) $\mathcal{G}$ is a q. coh. $\mathcal{O}_Y$-mod. $\Rightarrow f^*\mathcal{G}$ is a q. coh $\mathcal{O}_X$-module

(ii) If X, Y are noetherian schemes then $\mathcal{G}$ is coherent $\Rightarrow f^*\mathcal{G}$ is coherent.

(iii) X is a noetherian scheme (or f is q. compact) then $\mathcal{F}$ q. coh. $\Rightarrow f_*\mathcal{F}$ q. coh. $\mathcal{O}_Y$-module.

(iv) If X and Y are finite type over a field, then f is projective.

$\mathcal{F}$ coherent $\Rightarrow f_*\mathcal{F}$ is coherent.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathbb{P}^n_Y \arrow[dl, "pr."] \\ Y & \end{tikzcd}$$

Higher direct image sheaves

$f : X \to Y$ be a map of schemes.

f_* is a functor from $\mathcal{O}_X$-modules to $\mathcal{O}_Y$-modules.

f_* is left exact $\left\{ \begin{array}{l} 0 \to \mathcal{F}' \to \mathcal{F} \text{ exact} \\ \Rightarrow 0 \to f_*\mathcal{F}' \to f_*\mathcal{F} \text{ exact.} \end{array} \right.$

$R^i f_*(-)$: right derived functors of f_*.

$\left(R^0 f_* \mathcal{F} = f_* \mathcal{F} \right)$

Proposition : $\forall i \geq 0$, $R^i f_* \mathcal{F}$ is the sheaf associated to the presheaf : $V \longmapsto H^i(f^{-1}(V); \mathcal{F}|_{f^{-1}(V)})$
($V \subseteq Y$ open)

Proposition: $X \xrightarrow{f} \operatorname{Spec} A$, X noetherian.

$$(R^i f_*)(\mathcal{F}) = \left(H^i(X, \mathcal{F})\right)^{\sim}$$

Theorem : $f : X \to Y$ be a projective morphism, X, Y finite type over a field, $\mathcal{F}$ coherent on X.

(i) $R^i f_* \mathcal{F}$ is coherent $\forall i \geq 0$

(ii) $R^i f_*(\mathcal{F}(m)) = 0 \quad i > 0, m \gg 0$

Projection formula : $f : X \to Y$, $\mathcal{F}$ $\mathcal{O}_X$-module, $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank.

$$R^i f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong R^i f_* \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E}$$

(Note that if $\mathcal{E}$ is free, i.e. $\mathcal{E} = \mathcal{O}_Y^{\oplus n}$, then

$f^*\mathcal{E} = \mathcal{O}_X^{\oplus n}$, since $f^*\mathcal{O}_Y = \mathcal{O}_X$.

In this case the LHS = RHS.

If $\mathcal{E}$ is locally free, we need to patch up the free $\mathcal{O}_Y(\mathcal{U})$-modules over the open cover of Y to get the above result.)

$$\begin{tikzcd} & X = \mathbb{P}^n_k \times \mathbb{P}^m_k \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathbb{P}^n_k & & \mathbb{P}^m_k \end{tikzcd}$$

Line bundles on X are

$$f^*\mathcal{O}_{\mathbb{P}^n}(a) \otimes g^*\mathcal{O}_{\mathbb{P}^m}(b)$$

$a, b \in \mathbb{Z}$

Denote this by

$$L = \mathcal{O}_{\mathbb{P}^n}(a) \boxtimes \mathcal{O}_{\mathbb{P}^m}(b)$$

$$R^i f_* L = R^i f_*\Big(\underbrace{f^*\mathcal{O}_{\mathbb{P}^n}(a)}_{\mathcal{E}} \otimes_{\mathcal{O}_X} \underbrace{g^*\mathcal{O}_{\mathbb{P}^m}(b)}_{\mathcal{F}}\Big)$$

proj. formula

$$= R^i f_*\big(g^*\mathcal{O}_{\mathbb{P}^m}(b)\big) \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(a)$$

$$\begin{tikzcd} X = \mathbb{P}^n_k \times \mathbb{P}^m_k \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\curvearrowright"] & \mathbb{P}^m_k \arrow[d, "v"] \\ \mathbb{P}^n_k \arrow[r, "u"] & \operatorname{Spec} k. \end{tikzcd}$$

$$R^i f_*\big(g^*\mathcal{O}_{\mathbb{P}^m}(b)\big) \cong u^*\big(R^i v_* \mathcal{O}_{\mathbb{P}^m}(b)\big)$$

$$\simeq u^*\big(H^i(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(b))^{\sim}\big)$$

(this is a sheaf on Spec k)

(& it is a v.s.)

$$\simeq \mathcal{O}_{\mathbb{P}^n} \otimes H^i(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(b))$$

$$R^i f_* L = \begin{cases} 0 & 1 \le i \le m-1, \ b \in \mathbb{Z} \\ \mathcal{O}_{\mathbb{P}^n}(a)^{\oplus ?} & i=0 \quad b \ge 0 \\ & i = m \end{cases}$$

$X = \mathrm{Proj}\, S$; L is an invertible sheaf on X.

Maps $X \to \mathbb{P}^n_k$ (morphisms of proj. schemes)

$$\longleftrightarrow \left\{ \begin{array}{l} \text{globally generated line bundles} \\ L \text{ on } X \text{ and } s_0, \dots, s_n \in H^0(X, L) \\ \text{that generate } L \end{array} \right\}$$

An invertible sheaf L on X gives the following data : • a morphism $Y \xrightarrow{f} X$ ($Y = \mathrm{Spec}\ \mathrm{Sym}\, L$)

• an affine open cover $\{\mathcal{U}_i\}$ of X, $\mathcal{U}_i = \mathrm{Spec}\, A_i$ such that $f^{-1}(\mathcal{U}_i) \overset{\psi_i}{\simeq} \mathbb{A}_{\mathcal{U}_i} = \mathrm{Spec}\, A_i[t]$ and $\psi_j \circ \psi_i^{-1}$ on any affine open subset $\mathrm{Spec}\, A = V \subseteq \mathcal{U}_i \cap \mathcal{U}_j$ is given by a linear automorphism θ of $A[t]$. $\{ t \mapsto at \text{ for } a \in A \}$

<u>Sections of L over $\mathcal{U}$:</u>

$$L(\mathcal{U}) = \{ \mathcal{U} \xrightarrow{s} Y \ / \ f \circ s = id_{\mathcal{U}} \}$$

$s \in \Gamma(X, L)$; $s : X \to Y$, for $x \in X$, $s(x) \in f^{-1}(x) \simeq \mathbb{A}^1_k$

$\therefore s : X \to k$ is a function

If $s_0, \dots, s_n \in \Gamma(X, L)$ then we can define

$$X \longrightarrow \mathbb{P}^N$$
$$x \longmapsto (s_0(x), \dots, s_n(x))$$

$$\left\{ \begin{array}{l} \text{The fact that } s_0, \dots, s_n \text{ generate } L \\ \qquad \Rightarrow (s_0(x), \dots, s_n(x)) \neq (0 : 0 \cdots : 0) \\ \qquad\qquad \text{for any } x. \end{array} \right\}$$