Krishna (Talk - 7)

Based on Józefiak - Pragacz - Weyman paper.

Note Title 12/22/2012

Let $X = \mathbb{P}^1$, $L = \mathcal{O}_{\mathbb{P}^1}(2)$, $(X_0^2, X_1 X_2, X_2^2)$ are sections which generate $L$.

$\mathcal{O}(2) : \mathbb{P}^1 \longrightarrow \mathbb{P}^2$

$(a, b) \longmapsto (a^2, ab, b^2)$

This map is an embedding.

In general, if we take $\mathcal{O}_{\mathbb{P}^1}(d)$, $d \geq 2$

$\mathcal{O}_{\mathbb{P}^1}(d) : \mathbb{P}^1 \to \mathbb{P}^d$

$(a, b) \longmapsto (a^d, a^{d-1}b, \ldots, ab^{d-1}, b^d)$

is the Veronese embedding

These are very ample line bundles on $\mathbb{P}^1$.

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Let $X \subseteq \mathbb{P}^2$ be a closed subvariety

$S = k[x_0, \ldots, x_r]$

$I_X \subseteq S$ ideal

$S_X := S/I_X$ is homogeneous co-ord ring of $X$.

$X \subseteq \mathbb{P}^r$ is called "projectively normal" if $S_X$ is normal.

$$S' = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(n))$$

Assume $X$ is normal & connected, then $S_X$ is a domain

Also assume that $X$ is not degenerate i.e. $X \nsubseteq \mathbb{P}^{r-1}$.

$S_X \subset S'$ and $S'$ is the integral closure of $S_X$.

$X \subseteq \mathbb{P}^r$ is projectively normal if $S_X = S'$.

normality is an intrinsic ppty of $X$ while "proj. normal" is a ppty that depends on the embedding of $X$ in $\mathbb{P}^r$

Suppose $S'$ is a f.g. graded $S$-module

$$0 \to \bigoplus S(-a_{mi}) \to \cdots \to \bigoplus S(-a_{0i}) \longrightarrow S' \to 0$$

for some integers $a_{ki}$

we write this as

$$0 \longrightarrow F_m \to \cdots \longrightarrow F_0 \to S' \longrightarrow 0.$$

$X \subseteq \mathbb{P}^r$ is projectively normal $\iff S_X = S' = S/I_X$

$\iff F_0 = S$

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<u>$N_p$ ppty</u> :

Let $p \geq 0$. $X$ has $N_0$ ppty if $X$ is projectively normal.

$X$ has $N_1$ ppty if it has $N_0$ and $a_{1i} = 2 \ \forall i$ i.e.

$$F_1 = \bigoplus S(-2)$$

$\vdots$

$X$ has $N_p$ ppty if it has $N_0, \dots, N_{p-1}$ and $a_{pi} = p+1 \ \forall i$ i.e.

$$F_p = \bigoplus S(-(p+1)).$$

Note that this is a ppty. of the embedding $X \hookrightarrow \mathbb{P}^r$.

$\therefore$ we will say the corresponding line bundle has the $N_p$ ppty.

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Example : $X = \mathbb{P}^1$, $L = \mathcal{O}_{\mathbb{P}^1}(2)$.

Consider $\mathbb{P}^1 \longrightarrow \mathbb{P}^2$.

$$(a,b) \longmapsto (a^2, ab, b^2)$$

$S = k[x_0, x_1, x_2]$

Then $I = (x_1^2 - x_0x_2)$

$S/I$ is normal, $\therefore \mathcal{O}_{\mathbb{P}^1}(2)$ has $N_0$ & $N_1$.

$$0 \to S(2) \to S \to S/I \to 0$$

Let $L = \mathcal{O}_{\mathbb{P}^1}(3)$, $\mathbb{P}^1 \to \mathbb{P}^3$

$$(a,b) \mapsto (a^3, a^2b, ab^2, b^3)$$

Then $I = (x_1^2 - x_0x_2, \ x_1x_2 - x_0x_3, \ x_2^2 - x_1x_3)$

$S/I$ is normal.

$$0 \to S(-3)^{\oplus 2} \to S(-2)^{\oplus 3} \to S \to S/I \to 0$$

$\therefore \mathcal{O}_{\mathbb{P}^1}(3)$ has $N_2$.

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Eagon-Northcott complex :

Let $X = \mathbb{P}^1 \longrightarrow \mathbb{P}^d$.

The ideal of this embedding is given by $2 \times 2$ minors of $\begin{bmatrix} x_0 & \cdots & x_{d-1} \\ x_1 & \cdots & x_d \end{bmatrix}$

Then $\mathcal{O}_{\mathbb{P}^1}(d)$ has $N_{d-1}$.

In general:

Conjecture (Ottaviani - Paoletti)

easy to check.

$$\mathcal{O}_{\mathbb{P}^n}(d) \text{ has } N_p \iff \begin{cases} d=1 \ \forall n,p; \ n=1 \ \forall d,p. \\ n=2,\ d=2 \quad \forall p \\ n \geq 3,\ d=2,\ p \leq 5 \\ n \geq 2,\ d \geq 3,\ p \leq 3d-3. \end{cases}$$

$\Rightarrow$ proved by Ottaviani - Paoletti.

$\Leftarrow$ • $\left.\begin{array}{l} n=2 \\ n=d=3 \end{array}\right\}$ M. Green

• $d=2$ Józefiak - Pragacz - Weyman.