Quantum Mechanics 3, Spring 2012 CMI

Problem set 12

Due by beginning of class on Monday Apr 9, 2012

Dirac equation

1. Consider the Dirac equation in a spherically symmetric electrostatic potential $V(r)$

$$i\hbar\frac{\partial\psi}{\partial t} = \left[c\vec{\alpha}\cdot\vec{p} + \beta mc^2 + V(r)\right]\psi \tag{1}$$

Suppose ψ, ϕ are states that evolve via the Dirac equation.

   (a) $\langle 5\rangle$ Find whether the matrix elements of the components of orbital angular momentum $\langle\psi|L_i|\phi\rangle$ are conserved in time.

   (b) $\langle 3\rangle$ Similarly, are the matrix elements of total angular momentum $J_i = L_i + S_i$ conserved in time?

2. Working with γ matrices. Consider the matrices $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.

   (a) $\langle 2\rangle$ Find the adjoints of σ^{ij} and of σ^{0i} and express them in terms of the $\sigma^{\mu\nu}$'s.

   (b) $\langle 3\rangle$ Find whether γ^0 commutes or anti-commutes with σ^{ij} and with σ^{0i} where $i, j = 1, 2, 3$.

   (c) $\langle 2\rangle$ Show that $[\gamma^\alpha, [\gamma^\mu, \gamma^\nu]] = 4(\eta^{\mu\alpha}\gamma^\nu - \eta^{\nu\alpha}\gamma^\mu)$.

3. Simple illustration of invariance of Minkowski metric under Lorentz transformations in curvilinear coordinates.

   (a) $\langle 2\rangle$ Express the metric of Minkowski space in spherical polar coordinates ct, r, θ, ϕ, i.e., find the metric tensor $g_{\mu\nu}(p)$, where μ, ν enumerate (ct, r, θ, ϕ) in that order and $p = (ct, r, \theta, \phi)$.

   (b) $\langle 3\rangle$ Suppose we make a Lorentz transformation, say a rotation by an angle ϕ_0 counter clockwise about the z-axis. This maps points $p \mapsto f(p) = (ct', r', \theta', \phi')$. Show that the metric is unchanged by this rotation, namely show that

$$g_{\alpha\beta}(f(p))\frac{\partial f^\alpha}{\partial p^\mu}\frac{\partial f^\beta}{\partial p^\nu} = g_{\mu\nu}(p) \tag{2}$$