Quantum Mechanics 2, Autumn 2011 CMI
Problem set 12
Due by beginning of class on Monday November 14, 2011
Scattering Theory

1. We wish to find the coefficients C_l appearing in the expansion of a plane wave as a linear combination of spherical waves:

$$e^{ikr\cos\theta} = \sum_l C_l(2l+1)P_l(\cos\theta)j_l(kr). \qquad (1)$$

   (a) If $\rho = kr$, **show that** C_l must satisfy the following identity for each $l = 0, 1, 2, \ldots$ and $\rho \geq 0$:

$$\int_{-1}^{1} e^{i\rho x} P_l(x)\, dx \;=\; 2\, C_l\, j_l(\rho) \quad \text{where} \;\; x = \cos\theta. \qquad (2)$$

   The above identity must in particular be true as $\rho \to 0$. So we will compare the leading behavior of both sides as $\rho \to 0$ to extract the constants C_l. For this we use the formulae

$$P_l(x) = \frac{1}{(2l)!!}\frac{d^l}{dx^l}(x^2-1)^l \quad \text{and} \quad j_l(\rho) \to \frac{\rho^l}{(2l+1)!!} \;\; \text{as} \;\; \rho \to 0. \qquad (3)$$

   (b) Express the LHS of (2) in terms of the l^{th} derivative of $e^{i\rho x}$.

   (c) Find n_o and show that

$$\frac{d^l}{dx^l}e^{i\rho x} = \sum_{n=n_o}^{\infty} \frac{i^n \rho^n}{(n-l)!} x^{n-l}. \qquad (4)$$

   (d) Find the leading behavior of $\frac{d^l}{dx^l}e^{i\rho x}$ for small ρ.

   (e) Use this to show the behavior of the LHS of (2) for small ρ is given by

$$\int_{-1}^{1} e^{i\rho x} P_l(x)\, dx \;\to\; \frac{(-1)^l i^l \rho^l}{(2l)!!} \int_{-1}^{1} (x^2-1)^l\, dx. \qquad (5)$$

   (f) Comparing with the behavior of the RHS, find C_l given that

$$\int_0^{\frac{\pi}{2}} \cos^{2l+1}\theta\, d\theta = \frac{(2l)!!}{(2l+1)!!}. \qquad (6)$$

2. Consider classical scattering with energy $E > V_o$ in a 1d square-well/barrier potential $V(x) = \mp V_o \theta(|x| < a)$ with $V_o > 0$. In the three cases (i) free particle (ii) repulsive and (iii) attractive potential find the times $T_{0,\pm}$ taken by the particle to go from $x = -L$ to $x = L$ where $L > a > 0$. Draw the potentials and energy level in all cases. By comparing $T_\pm$ with T_0 define the *time-delay* τ and find it for the repulsive and attractive cases in the limit $L \to \infty$. Which one has a delayed arrival and which one an advanced arrival time? Give a qualitative explanation of the results.

3. Consider low wave number ($ka \ll 1$) scattering against a finite spherical barrier $V_o\theta(r < a)$ with $V_o > 0$. Assuming the S-wave phase shift $\delta_0 \ll 1$, find the S-wave scattering cross-section σ_0 as a function of the incident energy E. What is σ_0 in the limit $V_o \gg E$?