Quantum Mechanics 1, Spring 2011 CMI

Problem set 3

Due by the beginning of class on Friday January 28, 2011

Waves and cavity radiation

1. Find a simpler expression (as the product of two traveling waves) for the superposition of three waves with slightly different wave numbers and frequencies of the form

$$\psi(x,t) = \psi_- + \psi_0 + \psi_+ = \sin((k-\delta k)x - (\omega-\delta\omega)t) + 2\sin(kx-\omega t) + \sin((k+\delta k)x - (\omega+\delta\omega)t) \quad (1)$$

   Hints: Combine them in a convenient order! $\langle 2 \rangle$

2. Find the intensity of the combined wave and plot it at a convenient time. Describe the resulting waveform. You may assume $\delta k << k$ and $\delta\omega << \omega$. $\langle 3 \rangle$

3. Explain how you can assign a pair of speeds to the above waveform. Find formulae for the speeds, give them appropriate names. $\langle 2 \rangle$

4. Explain how to assign a reasonable width to the constituents of the above waveform and find a formula for the width. Are the constituents more or less sharply localized than when we combine just two of the waves? $\langle 3 \rangle$

5. How many waves do you think need to be combined to form a wave packet whose intensity decays away from a single region of localization. $\langle 2 \rangle$

6. What is the definition of the spectral radiance $u(\nu)$? Work out the dimensions of the Raleigh-Jeans and Planck distributions $u(\nu)$ and check that they have the dimensions appropriate to the above definition. $\langle 2 \rangle$

$$u_{RJ}(\nu) = \frac{8\pi\nu^2}{c^3} kT, \qquad u_P(\nu) = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{e^{h\nu/kT} - 1} \quad (2)$$

7. Wien's empirical law $\lambda_{max} T = \kappa$ states that the wavelength at which a body radiates maximally is inversely proportional to the temperature. Obtain this law from Planck's distribution for the spectral radiance $u_P(\nu)$ and find the constant κ. $\langle 5 \rangle$

8. Consider standing waves of the form $\sin(k_1 x)\sin(k_2 y)e^{i\omega t}$ in a two-dimensional square cavity of side L ($0 \le x \le L$, $0 \le y \le L$). Assuming the waves vanish on the boundary, express the allowed values of k_1 and k_2 in terms of appropriate non-negative integers n_1, n_2 and find the relation between $k = \sqrt{k_1^2 + k_2^2}$ and $n = \sqrt{n_1^2 + n_2^2}$. $\langle 1.5 \rangle$

9. What is the relation between n and the frequency ν? $\langle 1 \rangle$

10. Find an approximate formula for the number of standing electromagnetic waves $g(\nu)d\nu$ in the above square in the frequency range $[\nu, \nu + d\nu]$ for ν large enough. Keep in mind that an electromagnetic wave in 2 spatial dimensions has only one polarization state. $\langle 2 \rangle$

11. Assigning an energy of kT to each standing wave in equilibrium at temperature T, find the analogue of the Raleigh-Jeans distribution for the above square cavity. $\langle .5 \rangle$

12. What is the total energy per unit area radiated across all frequencies? Is there an ultra-violet catastrophe in this planar case? $\langle 1 \rangle$