Problems on Quantum Mechanics (Module 2)

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Classical Mechanics

1. How many degrees of freedom do the following systems have? (a) One point particle moving on a circle (b) Two point particles moving on a circle (c) Three point particles moving in a lecture hall (d) an irregularly shaped stone moving in the air above the playground.

2. Find Hamilton's equations for a particle in a double well potential with Hamiltonian

$$H = \frac{p^2}{2m} + g(x^2 - a^2)^2, \quad m, g, a > 0. \tag{1}$$

Check that they reduce to Newton's second order equation.

3. Check that the equations of motion $\dot{f}(x,p) = \{f(x,p), H\}$ reduce to Hamilton's equations for $f = x$ and $f = p$ for the above particle.

4. Find all static solutions to Hamilton's equations and classify them as stable/unstable to small perturbations.

5. Draw a phase portrait for the above particle in a double well potential.

Quantum Mechanics

6. Consider the Pauli matrices in the standard basis. Now $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Find $\sigma_2^\dagger$. Show that σ_2 is hermitian and unitary.

7. What is σ_2^2? What is the characteristic equation for σ_2?

8. Find a simple formula (in terms of trigonometric functions) for $e^{i\theta\sigma_2/2}$ by summing the exponential series.

9. What are the eigenvalues of σ_2? Find the corresponding eigenvectors v_+ and v_-.

10. Show that v_+ and v_- satisfy the completeness relation $v_+ v_+^\dagger + v_- v_-^\dagger = |v_+\rangle\langle v_+| + |v_-\rangle\langle v_-| = I_{2\times 2}$.

11. Show that the eigenvalue λ of an operator A may be interpreted as the expectation value of A in the corresponding eigenstate.

12. Show that the commutator of two hermitian operators is anti-hermitian.

13. Suppose a is any operator on a Hilbert space and let $H_1 = a^\dagger a$ and $H_2 = a a^\dagger$. Check that H_1 and H_2 are hermitian. Show that expectation values of H_1 and H_2 in all (non-zero) states are non-negative. We say that $aa^\dagger$ and $a^\dagger a$ are positive (more precisely non-negative) operators. Hint: recall the definition of the norm of a vector.

14. Argue that the expectation value of kinetic energy $T = p^2/2m$ is nonnegative in any state. Also argue that the expectation value of energy $T + \frac{1}{2}m\omega^2 x^2$ of a simple harmonic oscillator is positive in any state.

15. Consider two finite dimensional matrices A, B. What is $\text{tr}\, [A, B]$?

16. Consider the Heisenberg commutation relations $[x,p] = i\hbar I$. Comment on the result of calculating the trace of either side.

17. Recall that $\int_{-\infty}^{\infty} e^{iky}\, dy = 2\pi\delta(k)$. Evaluate the integral $\int_{-\infty}^{\infty} y\, e^{iky}\, dy$.

18. Find the matrix element of the position operator between momentum eigenstates $\langle k|x|k'\rangle$. Hint: $\langle x|k\rangle = e^{ikx}$ and use the completeness relation.

19. Find the matrix element of the momentum operator p between position eigenstates $\langle x|p|y\rangle$.

20. Given two observables (hermitian operators) A, B with $[A,B] = iC$ one may show the uncertainty inequality $(\Delta A)^2_\psi (\Delta B)^2_\psi \geq \frac{1}{4}\langle C\rangle^2_\psi$ for any (unit norm) state ψ. Say how $(\Delta A)_\psi$ is defined. Here $\langle C\rangle_\psi$ is the expectation value of C in the state ψ. Consider the angular momentum observables that satisfy $[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$. Apply the uncertainty inequality to $A = L_x$ and $B = L_y$ and comment on the result. Is there a state where the product of uncertainties can vanish? Contrast this with the case $A = x, B = p$.

21. Consider the Schrödinger equation for a particle in a 1d real potential $V(x)$:

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi(x). \tag{2}$$

Find the equation satisfied by the complex conjugate wave function.

22. Derive a local conservation law for probability in 1D: $\frac{\partial P}{\partial t} + \frac{\partial j}{\partial x} = 0$ where $P = |\psi(x,t)|^2$. What is the probability current $j(x,t)$?

23. Show that $\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$.

24. Show that the Gaussian wave packet $\psi(x) = Ae^{-\frac{x^2}{4a^2}}$ has unit norm if we take $A = \frac{1}{\sqrt{a}(2\pi)^{1/4}}$. What is the mean position and mean momentum of a particle in this state?

25. Consider a quantum mechanical particle moving in a potential $V(x)$ in one dimension. Its state evolves according to the Schrodinger equation. Use the Schrödinger equation to calculate the time evolution of the mean momentum of the particle, $\frac{d\langle p\rangle}{dt}$ and express the answer in terms of the expectation values of other familiar quantities. Recall that $\langle 4\rangle$

$$\langle p\rangle = \int dx\; \psi^*(x)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x). \tag{3}$$

26. Suppose the wave function of a particle in one dimension is bounded and decays like $\psi(x) \sim \frac{1}{|x|^\alpha}$ as $|x| \to \pm\infty$ for some power $\alpha > 0$. How small can the power α be and still ensure that the wave function has a finite norm? $\langle 3\rangle$

27. Evaluate the one-dimensional gaussian integral I_1 in closed form.

$$I_1 = \int_{-\infty}^{\infty} dx\; e^{-x^2} \tag{4}$$

Hint: Consider the Gaussian integral I_2, and evaluate it by transforming to polar coordinates on the x-y plane. How is I_2 related to I_1? $\langle 3\rangle$

$$I_2 = \iint dx\; dy\; e^{-x^2-y^2} \tag{5}$$

28. Suppose $\tilde{f}(k)$ is the Fourier transform of $f(x)$,

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x) \tag{6}$$

Find the Fourier transform $\tilde{g}(k)$ of the function $g(x) = xf(x)$. In other words, express $\tilde{g}(k)$ in terms of $\tilde{f}(k)$. $\langle 2.5 \rangle$

29. Multiplication by x in position space is represented by what operation in k-space? $\langle 1 \rangle$

30. Let u, v be vectors and A an operator on $\mathcal{H}$. Simplify (a) $\langle iAv|u\rangle$, (b) $\langle u|iAv\rangle$ and (c) $\langle iAv|iu\rangle$ for the standard L^2 inner product. In other words, pull out the i's properly! $\langle 2 \rangle$

31. Based on the reality of average experimental measurements of observables, we defined a hermitian operator A as one with real expectation values in every state ψ, i.e., ($\langle\psi|A\psi\rangle \in \mathbb{R}$). Say why this is the same as $\langle 2 \rangle$

$$\langle\psi|A\psi\rangle = \langle A\psi|\psi\rangle. \tag{7}$$

32. A more conventional definition of hermiticity is that the matrix elements of A satisfy

$$\langle u|Av\rangle = \langle Au|v\rangle \tag{8}$$

for any pair of states u, v. Say why this is the same as $A_{uv} = (A_{vu})^*$. $\langle 2 \rangle$

33. Now suppose A satisfies (7). We wish to show that it also satisfies (8). To show this, we put $\psi = u + v$ and $\psi = u + iv$ in (7) and add the two resulting equations. Show that this reduces to $A_{uv} = (A_{vu})^*$. Thus the reality of expectation values in all states implies that A is hermitian in the conventional sense. The converse is much simpler. $\langle 5 \rangle$

34. Consider a particle in a (real) potential $V(x)$. Suppose $\psi(x)$ is a solution of the time-independent Schrödinger equation with (real) energy eigenvalue E. Find another wave function that has the same eigenvalue E. When are the two eigenfunctions the same? $\langle 2 \rangle$

35. Use the result of the previous problem to show that for any energy eigenvalue E, one can always find a corresponding real eigenfunction of the hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$. This feature is because H is not just hermitian but also real-symmetric. $\langle 3 \rangle$

36. If P, Q are hermitian, what can you say about the commutator $[P, Q]$? Can $[P, Q]$ be an observable? $\langle 2 \rangle$

37. It is possible to show (using Cauchy-Schwarz) that for position and momentum x and p,

$$(\Delta x)^2\,(\Delta p)^2 \geq -\frac{1}{4}\langle [x, p]\rangle_\psi^2 \tag{9}$$

where $(\Delta x)^2 = \langle x^2\rangle_\psi - \langle x\rangle_\psi^2$ is the variance of x in the state ψ and similarly for Δp. Show that this reduces to the Heisenberg uncertainty principle. $\langle 2 \rangle$

38. Consider the momentum operator $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ of a particle that is constrained to move in the interval $[-1, 1]$. Give a convenient choice of boundary condition for $\psi(\pm 1)$ that ensures that $\hat{p}$ is hermitian. Give the physical meaning of the boundary condition that you propose. $\langle 5 \rangle$

39. Consider a free particle that moves in the interval $0 \le x \le L$. Unlike the square well problem, here we will impose *periodic boundary conditions*: all states must satisfy $\psi(0) = \psi(L)$ and $\psi'(0) = \psi'(L)$[1].

   (a) Does $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ have any eigenfunctions satisfying periodic boundary conditions? If so find the orthonormal eigenfunctions and corresponding eigenvalues. $\langle 5 \rangle$

   (b) Do you think one could make measurements of $\hat{p}$ with arbitrary precision (in principle)? If such a measurement is made, what are the possible values of momentum that one might obtain? What is the state of the particle after one such measurement? $\langle 3 \rangle$

   (c) For the above particle with hamiltonian $\hat{H} = \frac{p^2}{2m}$, find the normalized energy eigenfunctions and eigenvalues. Work with exponentials rather than trigonometric functions here. $\langle 5 \rangle$

   (d) Can $\hat{H}$ and $\hat{p}$ be simultaneously measured with arbitrary accuracy? Why? $\langle 1 \rangle$

   (e) What is the ground state wave function and energy? Plot the absolute square of the ground state wave function in position space. Where along the circle (or the interval $[0, L]$) is the particle most likely to be found in the ground state? $\langle 3 \rangle$

   (f) How many linearly independent eigenfunctions are there at each energy level? $\langle 2 \rangle$

   (g) The potential here is zero, which is real. So find real energy eigenfunctions (in the position basis) at each energy level. $\langle 3 \rangle$

   (h) The potential here is zero, which is an even function of x about the point $x = L/2$. Find the energy eigenfunctions of definite parity (even or odd about $x = L/2$) at each energy level. Is there both an even and an odd state at each energy level? $\langle 3 \rangle$

40. Free particle gaussian wave packet and harmonic oscillator.

   (a) Recall that the gaussian wave packet

$$\psi(x, t = 0) = Ae^{ik_0 x} e^{-\frac{x^2}{4a^2}}, \quad A^2 = \frac{1}{a\sqrt{2\pi}}. \tag{10}$$

   has mean momentum $\langle p \rangle = \hbar k_0$ at $t = 0$. Write down $\tilde{\psi}(k, t = 0)$ and then obtain $\tilde{\psi}(k, t)$ in the energy/momentum basis. $\langle 3 \rangle$

   (b) Find $\langle p \rangle$ at $t > 0$. $\langle p \rangle$ is most easily calculated in the momentum basis. $\langle 4 \rangle$

   (c) Calculate $\langle \hat{x} \rangle$ at time t in the above gaussian wave packet. Since $\tilde{\psi}(k, t)$ is known, it is good to work in the momentum basis. So you need to know how $\hat{x}$ acts in k-space. This was worked out in problem set 6: $\hat{x} = i\frac{\partial}{\partial k}$. Hint: In working out the integrals, exploit the fact that integrals of odd functions on even intervals vanish. $\langle 9 \rangle$

   (d) Do the obtained mean values satisfy Ehrenfest's principle $m\frac{\partial \langle x \rangle}{\partial t} = \langle p \rangle$ at all times? $\langle 2 \rangle$

[1] A physical realization is a particle moving on a circle: x is the coordinate along the circumference (or the polar angle ϕ), so the coordinates $x = 0$ and $x = L$ represent the same physical point (the angles $\phi = 0, 2\pi$). In this physical realization, the momentum we refer to is really the component of angular momentum L_ϕ where ϕ is the polar angle in the plane of the circle.

(e) Find the probability for a particle in the ground state $\frac{\sqrt{\beta}}{\pi^{1/4}} e^{-\beta^2 x^2/2}$ of a harmonic oscillator potential $\frac{1}{2} m\omega^2 x^2$, to be found outside its classically allowed region. Express this probability as an integral over dimensionless variables. Does it depend on m or ω? Here $\beta = \sqrt{\frac{m\omega}{\hbar}}$. $\langle 5 \rangle$

(f) Find the numerical value of this probability. You may use $\int_1^\infty d\xi \, e^{-\xi^2} \approx 0.14$. $\langle 2 \rangle$