Problems: Introduction to Fields and their Quantization
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1. Show that the Euler-Lagrange equations following from the Lagrangian density
$$\mathcal{L} = \frac{\rho}{2} u_t^2 - \frac{\tau}{2} |\nabla u|^2 \quad (1)$$
lead to the 3d wave equation $\rho u_{tt} = \tau \nabla^2 u$.

2. Show that
$$\mathcal{L} = \frac{1}{2c^2} (\partial_t \phi)^2 - \frac{1}{2} |\nabla \phi|^2 - \frac{\phi^2}{2\lambda^2} \quad (2)$$
furnishes a Lagrangian density for the Klein-Gordon equation $(1/c^2)\phi_{tt} = \nabla^2 \phi - (1/\lambda^2)\phi$.

3. Verify that a Lagrangian density for the self-interacting scalar field equation $\frac{1}{c^2}\partial_t^2 \phi - \nabla^2 \phi + \frac{1}{\lambda^2}\phi + g\phi^3 = 0$ is given by
$$\mathcal{L} = \frac{1}{2c^2} (\partial_t \phi)^2 - \frac{1}{2} |\nabla \phi|^2 - \frac{\phi^2}{2\lambda^2} - \frac{g}{4}\phi^4. \quad (3)$$

4. Verify that Hamilton's equations $u_t = \{u, H\}$ and $\pi_t = \{\pi, H\}$ following from the Hamiltonian
$$H[u, \pi] = \int_0^\ell \left(\frac{\pi^2}{2\rho} + \frac{\tau}{2} u_x^2 \right) dx \quad (4)$$
and the Poisson brackets
$$\{u(x,t), \pi(x',t)\} = \delta(x - x'), \quad \{u(x,t), u(x',t)\} = \{\pi(x,t), \pi(x',t)\} = 0 \quad (5)$$
are equivalent to the wave equation $\rho u_{tt} = \tau u_{xx}$.

5. Show that the energy
$$H = \int \left(\beta |\partial_x \psi|^2 + \frac{1}{2}\kappa |\psi(x)|^4 \right) dx \quad (6)$$
is conserved with respect to the time evolution defined by the nonlinear Schrödinger equation
$$i\partial_t \psi = -\alpha\beta \partial_x^2 \psi + \alpha\kappa |\psi|^2 \psi. \quad (7)$$

6. Use the Hamiltonian
$$H = \int \left(\beta |\partial_x \psi|^2 + \frac{1}{2}\kappa |\psi(x)|^4 \right) dx \quad (8)$$

and Poisson brackets

$$\{\psi(x), \psi^*(x')\} = -i\alpha\delta(x - x'), \quad \{\psi(x), \psi(x')\} = \{\psi^*(x), \psi^*(x')\} = 0, \tag{9}$$

to show that Hamilton's equation for ψ^* gives the complex conjugate of the NLSE:

$$\dot{\psi}^* = -i\alpha\beta\psi^{*\prime\prime} + i\alpha\kappa|\psi|^2\psi^*. \tag{10}$$

7. Verify that $\{N, H\} = 0$ for the nonlinear Schrödinger field with $N = \int \psi^*\psi dx$. This is expected from the conservation of N.
8. Verify that $\{P, H\} = 0$ for the nonlinear Schrödinger field with $P = i\alpha\beta \int \psi^*\psi'\, dx$. This is expected from the conservation of the field velocity P.
9. Show that the Euler-Lagrange equation for ψ following from the Lagrangian

$$L = \int \left[\frac{i}{\alpha}\psi^*\dot{\psi} + \beta\psi^*\partial_x^2\psi - \frac{1}{2}\kappa\psi^*\psi^*\psi\psi \right] dx \tag{11}$$

leads to the complex conjugate NLSE.

10. Argue why the number operator $\boldsymbol{N} = \int \boldsymbol{\phi}^\dagger(x)\boldsymbol{\phi}(x)\, dx$ is hermitian and positive definite.
11. Show that the Hamiltonian operator of the nonlinear Schrödinger field commutes with the number operator $[\boldsymbol{H}, \boldsymbol{N}] = 0$. Here $\boldsymbol{N} = \int \boldsymbol{\phi}^\dagger(x)\boldsymbol{\phi}(x)\, dx$ and

$$\boldsymbol{H} = -\beta \int \boldsymbol{\phi}^\dagger(x)\partial_x^2\boldsymbol{\phi}(x)\, dx + \frac{\kappa\hbar\alpha}{2} \int \boldsymbol{\phi}^\dagger(x)\boldsymbol{\phi}^\dagger(x)\boldsymbol{\phi}(x)\boldsymbol{\phi}(x)\, dx. \tag{12}$$

Hint: Use the canonical commutation relations

$$[\boldsymbol{\phi}(x), \boldsymbol{\phi}^\dagger(y)] = \delta(x - y), \quad [\boldsymbol{\phi}(x), \boldsymbol{\phi}(y)] = [\boldsymbol{\phi}^\dagger(x), \boldsymbol{\phi}^\dagger(y)] = 0. \tag{13}$$

12. Use the canonical commutation relations and definition of the number operator $\boldsymbol{N}$ to show that

$$[\boldsymbol{N}, \boldsymbol{\phi}(x)] = -\boldsymbol{\phi}(x) \quad \text{and} \quad [\boldsymbol{N}, \boldsymbol{\phi}^\dagger(x)] = \boldsymbol{\phi}^\dagger(x). \tag{14}$$

13. Use the canonical commutation relations to show that

$$\nu_3 = \int dx_3 \boldsymbol{\phi}^\dagger(x_3)\boldsymbol{N}^2\boldsymbol{\phi}(x_3) = \boldsymbol{N}(\boldsymbol{N} - I)^2. \tag{15}$$

14. Try to show by induction that

$$\int d^N x\, \boldsymbol{\phi}^\dagger(x_N)\cdots\boldsymbol{\phi}^\dagger(x_1)\boldsymbol{\phi}(x_1)\cdots\boldsymbol{\phi}(x_N) = \boldsymbol{N}(\boldsymbol{N}-I)(\boldsymbol{N}-2I)\cdots(\boldsymbol{N}-(N-1)I)I. \tag{16}$$

15. Use induction to show that

$$[\boldsymbol{\phi}_1\boldsymbol{\phi}_2\cdots\boldsymbol{\phi}_N, \boldsymbol{H}] = \sum_{j=1}^{N} \boldsymbol{\phi}_1\cdots\boldsymbol{\phi}_{j-1}[\boldsymbol{\phi}_j, \boldsymbol{H}]\boldsymbol{\phi}_{j+1}\cdots\boldsymbol{\phi}_N \tag{17}$$

for any operators $\boldsymbol{\phi}_1, \cdots, \boldsymbol{\phi}_N, H$.

16. Establish the commutator relation

$$[\boldsymbol{\phi}(x), \boldsymbol{n}(y)] = \delta(x-y)\boldsymbol{\phi}(y), \tag{18}$$

where $\boldsymbol{n}(y) = \boldsymbol{\phi}^{\dagger}(y)\boldsymbol{\phi}(y)$.