QFT Anirban Basu 20 Aug 2018 Lect 1.

Symmetries Poincare, Galilean, scaling, Conformal, Lorentz SUSY. Lagrangian description (admitting) & admitting weak coupling esp.

Classical Mech $x(t)$ $p(t)$ $\mapsto$ Field Th $\phi(x^\mu)$ $\pi(x^\mu)$

$\eta_{\mu\nu} = \begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$. $c = \hbar = 1$.

<u>Scalars</u> $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V[\phi(x)] = \frac{1}{2}\dot\phi^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)$.

1. Free ~~V=0~~ $V = \frac{1}{2}m^2\phi^2$.
2. Higgs Englert Brout $V = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$.
3. $m^2 \to -m^2$

<u>Spin $\frac{1}{2}$</u>: Dirac field. $\mathcal{L} = \bar\Psi(i\gamma^\mu\partial_\mu - m)\Psi$ free.

Interactions: $\mathcal{L} = \bar\Psi(i\gamma^\mu\partial_\mu - m)\Psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + g\bar\Psi\Psi\phi$.

Yukawa.

<u>Spin 1</u>: Maxwell & YM theory $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$; $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Free theory. Interactions: $\mathcal{L} = \bar\Psi(i\gamma^\mu D_\mu - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ QED

$D_\mu = \partial_\mu + ieA_\mu$ Local symm.

<u>Dynamics</u>: $S = \int d^4x\,\mathcal{L}$ $\mathcal{L}$ local, S real. $\mathcal{L}$ no more than 2 der.

Action principle & EL eqns. Neumann BC $\int d\sigma^\mu(\partial_\mu\phi)\delta\phi = 0$

if $\partial_\mu\phi\, d\sigma^\mu = 0$ normal deriv vanishes on boundary $\partial\Omega$

<u>Infinitesimal changes</u>: $\delta x^\mu = x'^\mu - x^\mu$; $\delta\phi \equiv \phi'(x) - \phi(x)$

$d^4x' = J\,d^4x$ $J = |\det(\frac{\partial x'^\mu}{\partial x^\nu})|$ $x'^\mu = x^\mu + \delta x^\mu$

$J = |\det(\delta^\mu_\nu + \partial_\nu\delta x^\mu)|$ $\det M = e^{\operatorname{tr}\log M}$ $M = I + \epsilon$

$\det(I+\epsilon) = \exp\operatorname{tr}\log(I+\epsilon) \hat= I + \operatorname{tr}\epsilon$

So $d^4x' = (1 + \partial_\mu\delta x^\mu)\,d^4x$

$\delta\mathcal{L} = \mathcal{L}(\phi'(x'), \partial'_\mu\phi'(x')) - \mathcal{L}(\phi(x), \partial_\mu\phi(x))$.

$= \delta x^\mu\partial_\mu\mathcal{L} + \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\underbrace{\delta\partial_\mu\phi}_{\partial_\mu\delta\phi} = \delta x^\mu\partial_\mu\mathcal{L} + \delta\phi\frac{\delta\mathcal{L}}{\delta\phi} + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi\right)$

$- \delta\phi\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\right) = \delta x^\mu\partial_\mu\mathcal{L} + \delta\phi\underbrace{\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\right]}_{0 \text{ if EL eqn sats}} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi\right)$

$\delta S = \underbrace{\int d^4x(\cancel{1+}\,\partial_\mu\delta x^\mu)\mathcal{L} + \int d^4x[\delta x^\mu\partial_\mu\mathcal{L}}_{\text{combine}} + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi\right)]$

$\delta S = \int d^4x\,\partial_\mu\left[\delta x^\mu\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi\right] \to$ for config sats EoM.

Eg $\delta x^\mu = \frac{\partial x^\mu}{\partial w^a}\delta w^a$ ↳ params $\delta\phi(x) = \frac{\partial \phi(x)}{\partial \omega^a}\delta w^a$ (2)

$\delta w^a \to$ global

use this above in $\delta S = \int d^4x\, \partial_\mu [\underbrace{\delta x^\mu}_{\frac{\partial x^\mu}{\partial w^a}\delta w^a} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \underbrace{\delta\phi}_{\frac{\partial\phi}{\partial w^a}\delta w^a}]$

to get

$$\delta S = \int d^4x\, \delta w^a\, \partial_\mu \left[\frac{\partial x^\mu}{\partial w^a}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\frac{\partial \phi}{\partial w^a}\right]$$

Now suppose we consider symm of action $\delta S = 0$ under this variation for arbit δw^a. Then we must have

$\partial_\mu J^{\mu a} = 0$ where $J^{\mu a} = \frac{\partial x^\mu}{\partial w^a}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\frac{\partial \phi}{\partial w^a}$. Cons curr for any a

Noether's Theorem. Every inf global symm transf there is a cons current.

Implications of Noether's Thm: $0 = \int dt \int d^3x\, \partial_\mu J^{\mu a} = 0$

$$0 = \int_{t_1}^{t_2} dt \int d^3x\, \partial_t (J^{0a}) + \underbrace{\int dt \int d^3x\, \partial_i J^{ia}}_{|\vec{x}| \to \infty}$$

take fields s.t. they vanish as $|\vec{x}| \to \infty$. Then

$$0 = \int d^3\vec{x}\, J^{0a}(\vec{x}, t_2) - \int d^3x\, J^{0a}(x, t_1) \quad \forall\, t_1, t_2.$$

$Q^a = \int d^3x\, J^{0a}(\vec{x}, t)$ is indep of time

Eg Scalar Fld: $S = \int d^4x [\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)]$ get Eom.

$\delta S = \int d^4x [\,\partial_\mu\phi\,\partial^\mu(\delta\phi) - V'(\phi)\delta\phi\,]$ integ by parts

$-\partial_\mu\partial^\mu\phi - V'(\phi) = 0 \Rightarrow \partial^2\phi = -V'(\phi)$.

Inf translations $x'^\mu = x^\mu + \epsilon^\mu$ $\delta x^\mu = \epsilon^\mu$

$\phi'(x') = \phi(x) \Rightarrow \phi'(x+\epsilon) = \phi(x) \simeq \phi'(x) + \epsilon^\mu \partial_\mu \underbrace{\phi'(x)}_{\text{diff from } \phi(x) \text{ by } O(\epsilon)}$

$\simeq \phi'(x) + \epsilon^\mu\partial_\mu\phi(x)$

so $\phi'(x) - \phi(x) = \delta\phi(x) = -\epsilon^\mu\partial_\mu\phi(x)$

$$\delta S = \int d^4x\, \partial_\mu\left[\delta x^\mu \mathcal{L} + \delta\phi \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right] = \int d^4x\, \partial_\mu\left[\epsilon^\mu\mathcal{L} - \epsilon^\nu\partial_\nu\phi\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\right]$$

$$\delta S = \int \epsilon^\nu \partial_\mu \left[\delta^\mu_\nu \mathcal{L} - \partial_\nu\phi\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\right] d^4x = 0 \quad \forall\, \epsilon^\nu$$

So $\partial_\mu T^{\mu\nu} = 0$ $T_{\mu\nu} = -\eta_{\mu\nu}\mathcal{L} + \partial_\nu\phi \underbrace{\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}}_{\partial_\mu\phi}$ = egy mom tensor Stress tensor

$T_{\mu\nu} = -\eta_{\mu\nu}\mathcal{L} + \partial_\mu\phi\,\partial_\nu\phi$

Belinfante

Can obtain symmetric $T_{\mu\nu}$ $T_{(\mu\nu)}$ Belifante construction Belifante tensor.

Now $\partial_\mu T^{\mu\nu} = -\partial^\nu \mathcal{L} + \underbrace{(\partial^2\phi)\partial^\nu\phi + \partial_\mu\phi\,\partial^\mu\partial_\nu\phi}$ (3)

$= -\partial^\nu(\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)) + \downarrow$

$= -\cancel{\partial_\mu\phi\,\partial^\mu\partial^\nu\phi} + V'(\phi)\partial^\nu\phi - \partial^2\phi\,\partial^\nu\phi + \cancel{\partial_\mu\phi\,\partial^\mu\partial^\nu\phi}$

$= \partial^\nu\phi\,(\partial^2\phi + V'(\phi)) = 0$

by EOM

Translations in time

Conserved charges: $P^\mu = \int d^3x\; T^{0\mu} = \int d^3x[-\mathcal{L}\eta^{\mu 0} + \dot\phi\,\partial^\mu\phi]$

So $P^0 = \int d^3x[-\mathcal{L} + \dot\phi^2] = \int d^3x[\;\dot\phi^2 - [\frac{1}{2}\dot\phi^2 - \frac{1}{2}|\nabla\phi|^2 - V(\phi)]]$

total egy $= \int d^3x\,\underbrace{(\frac{1}{2}\dot\phi^2 + \frac{1}{2}|\nabla\phi|^2 + V'(\phi))}_{\text{egy density}}$

Ground state: sum of squares $\dot\phi = 0,\ \nabla\phi = 0,\ V'(\phi) = 0.$

So $\phi(x^\mu) = \phi_0$ const s.t $V'(\phi_0) = 0$.

space ~~Lorentz~~ transformations $P^i = \int d^3x\; T^{0i} = \int d^3x\;\dot\phi\,(-\partial_i\phi)$

$\vec{P} = \int d^3x\;\dot\phi\,\nabla\phi$ conj mom $\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \dot\phi$

So $\vec{P} = \int d^3x\;\underbrace{\pi(x)\;\nabla\phi(x)}$ is mom density.

Infinitesimal Lorentz Transf: $x'^\mu = \Lambda^\mu{}_\nu x^\nu$

L.T $x^\mu x_\mu$ invariant. $x'^\mu x'_\mu = x^\mu x_\mu \Rightarrow \eta_{\mu\nu}x'^\mu x'^\nu = x^\mu x^\nu$

$\Rightarrow \eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta x^\alpha x^\beta = \eta_{\alpha\beta}x^\alpha x^\beta \Rightarrow \boxed{\eta_{\alpha\beta} = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\,\eta_{\mu\nu}.}$

Inf. Lorentz tr: $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$

So $\eta_{\alpha\beta} = \eta_{\mu\nu}(\delta^\mu{}_\alpha + \omega^\mu{}_\alpha)(\delta^\nu{}_\beta + \omega^\nu{}_\beta)$

$= \eta_{\alpha\beta} + \omega_{\beta\alpha} + \omega_{\alpha\beta} + O(\omega^2) \Rightarrow \omega_{\alpha\beta} = -\omega_{\beta\alpha}$

So $\delta x^\mu = \omega^\mu{}_\nu x^\nu$ $\phi'(x^\mu + \omega^\mu{}_\nu x^\nu) = \phi(x)$ scalar

So $\phi(x) = \phi'(x^\mu) + \omega^\mu{}_\nu x^\nu\partial_\mu\phi(x) = \phi'(x^\mu) - \omega^{\mu\nu}x_\mu\partial_\nu\phi(x)$

$\Rightarrow \delta\phi = +\,\omega^{\mu\nu}x_\mu\partial_\nu\phi(x)$

$\delta S = \int d^4x\,\partial_\mu[\omega^\mu{}_\alpha x^\alpha\mathcal{L} + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\omega^{\alpha\beta}x_\alpha\partial_\beta\phi]$

$= \int d^4x\,\frac{1}{2}\partial_\mu[\underbrace{(\delta^\mu{}_\alpha x_\beta - \delta^\mu{}_\beta x_\alpha)\mathcal{L} + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}(x_\alpha\partial_\beta - x_\beta\partial_\alpha)\phi}_{-J^\mu{}_{\alpha\beta}}]$

$\Rightarrow -J_{\mu\alpha\beta} = (\eta_{\mu\alpha}x_\beta - \eta_{\mu\beta}x_\alpha)\mathcal{L} + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}(x_\alpha\partial_\beta - x_\beta\partial_\alpha)\phi$

4

$$J_{\mu\alpha\beta} = x_\alpha \underbrace{\left(-\eta_{\mu\beta}\mathcal{L} + \partial_\beta\phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\right)}_{T_{\mu\beta}} + x_\beta \underbrace{\left(\eta_{\mu\alpha}\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\partial_\alpha\phi\right)}_{-T_{\mu\alpha}}$$

$$\boxed{J^\mu{}_{\alpha\beta} = T^\mu{}_\alpha x_\beta - T^\mu{}_\beta x_\alpha}$$ cons. ~~charges~~ currents

$$\boxed{M_{\alpha\beta} = \int d^3x \, J^0{}_{\alpha\beta}}$$ cons charges $M_{ij} = \int d^3x \, (\mathcal{P}_i x_j - \mathcal{P}_j x_i)$

$$P_i = \int d^3x \, T^0{}_i = \int d^3x \, \mathcal{P}_i$$

$$J_i = \frac{1}{2}\epsilon_{ijk} M_{jk} = \epsilon_{ijk}\int d^3x \, \mathcal{P}_j x_k$$

$$\vec{J} = -\int d^3x \, (\vec{r} \times \vec{\mathcal{P}}) \, .$$

Boosts $M_{0i} = K_i = \int d^3x \, [T_{00} x_i - T_{0i} x_0] = \int d^3x \, [x_i \mathcal{H} + t\mathcal{P}_i]$

$$\vec{K} = \int d^3x \, (\vec{x}\mathcal{H} - t\vec{\mathcal{P}})$$

Next EM field. get stress tensor using Belifante.

Belinfante

$S = \int d^4x\, \mathcal{L} \qquad \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \qquad \delta(d^4x) = \partial_\mu \delta x^\mu \, d^4x$

$\delta S = \int (\partial_\mu \delta x^\mu) \mathcal{L}\, d^4x + \int d^4x\, \delta\mathcal{L}$; $\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\delta \partial_\mu \phi + \partial_\mu \mathcal{L}\, \delta x^\mu$.

$\delta\mathcal{L} = \cancel{\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi} + \partial_\mu\left[\delta\phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\right] - \cancel{\delta\phi\, \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial_\mu \phi}\right)} + \delta x^\mu \partial_\mu \mathcal{L}$

$\Rightarrow \delta S = \int d^4x \left[(\partial_\mu \delta x^\mu)\mathcal{L} + \delta x^\mu \partial_\mu \mathcal{L} - \partial_\mu\left(\delta\phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\right)\right]$

$\Rightarrow \delta S = \int d^4x\, \partial_\mu \left[\delta x^\mu \mathcal{L} + \delta\phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\right]$

Now take $\delta x^\mu = \frac{\partial x^\mu}{\partial \omega^a}\delta\omega^a$, $\delta\phi = \frac{\partial \phi}{\partial \omega^a}\delta\omega^a$ ω^a global

So $\delta S = \int d^4x\, \delta\omega^a \partial_\mu \left[\frac{\partial x^\mu}{\partial \omega^a}\mathcal{L} + \frac{\partial \phi}{\partial \omega^a}\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\right]$

$\delta\omega^a$ are arbitrary so $\partial_\mu J^\mu = 0$

Dimensional analysis

$[\phi] = \frac{d-2}{2}$ d dim scalar

In 4d: $\int d^4x \left\{ \frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4!}\phi^4 + \frac{g'}{\Lambda^4}(\partial_\mu\phi)^2(\partial_\nu\phi)^2 \right.$ Λ = UV cutoff

$\frac{g'}{\Lambda^4} \to$ neg. mass dim. irrelevant interactions. They can also violate causality EOM will have higher than 2nd time der.

Lorentz transformations & Noether current

$J_{\mu\alpha\beta} = T_{\mu\alpha} x_\beta - T_{\mu\beta} x_\alpha \Rightarrow \partial^\mu J_{\mu\alpha\beta} = T_{\beta\alpha} - T_{\alpha\beta}$

So $J_{\mu\alpha\beta}$ is ~~symm~~ conserved if $T_{\alpha\beta} = T_{\beta\alpha}$. ie stress tensor should be symm. But what if T is not symm. Belinfante const $T_{\alpha\beta} \to \tilde{T}_{\alpha\beta} = \tilde{T}_{\beta\alpha}$ by adding a term.

Maxwell Field

$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$A_\mu = (\phi, \vec{A})$ gauge tr $A_\mu \to A_\mu + \partial_\mu \lambda$ takes $F_{\mu\nu} \to F_{\mu\nu}$

$F^{0i} = E^i$, $F^{ij} = \epsilon^{ijk}B^k$

Apply Noether for Poincare. $A'_\mu(x) = \frac{\partial x^\alpha}{\partial x'^\mu}A_\alpha(x)$ $\to$ tr as a vector

translation: $x'^\mu = x^\mu + a^\mu$, $x^\mu = x'^\mu - a^\mu \Rightarrow \frac{\partial x^\mu}{\partial x'^\nu} = \delta^\mu_\nu$

$\boxed{\delta x^\mu = a^\mu}$

$A'_\mu(x+a) = A_\mu(x) = A'_\mu(x) + a^\alpha \partial_\alpha A_\mu(x)$ 6

So $\boxed{\delta A_\mu(x) = -a^\alpha \partial_\alpha A_\mu(x)}$

So $\delta S = \int d^4x\, \partial_\mu \left[\delta x^\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \partial_\mu A^\nu} \delta A^\nu \right] = \int d^4x\, \partial_\mu \left[a^\mu \mathcal{L} - F^{\mu\nu} a^\alpha \partial_\alpha A_\nu \right]$

need -

Now $\frac{\partial \mathcal{L}}{\partial \partial_\alpha A_\beta} = -F_{\alpha\beta}$

So $\delta S = \int d^4x\, a^\lambda \partial_\mu \left[\delta^\mu_\lambda \mathcal{L} + F^{\mu\nu} \partial_\lambda A_\nu \right]$

So we have a conserved stress tensor $\partial_\mu T^{\mu\nu} = 0$.

$\boxed{T_{\mu\nu} = \eta_{\mu\nu} \mathcal{L} + F_\mu{}^\alpha \partial_\nu A_\alpha}$ → not symm (2nd term) add & subtract $\partial_\alpha A_\nu$ } also not gauge inv.

$T_{\mu\nu} = \eta_{\mu\nu} \mathcal{L} + F_\mu{}^\alpha (\partial_\nu A_\alpha - \partial_\alpha A_\nu) + F_\mu{}^\alpha \partial_\alpha A_\nu$

$= \underbrace{(\eta_{\mu\nu} \mathcal{L} + F_\mu{}^\alpha F_{\nu\alpha})}_{\tilde{T}_{\mu\nu} = \tilde{T}_{\nu\mu}} + F_\mu{}^\alpha \partial_\alpha A_\nu$

What is diff ?

So $\boxed{\tilde{T}_{\mu\nu} = -\frac{1}{4} \eta_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} + F_\mu{}^\lambda F_{\nu\lambda}}$

$\tilde{T}_{\mu\nu} = T_{\mu\nu} - F_\mu{}^\alpha (\partial_\alpha A_\nu)$ Now $\tilde{P}_\mu = \int d^3x\, \tilde{T}_{0\mu} = \int d^3x \left[T_{0\mu} - F_0{}^\alpha (\partial_\alpha A_\mu) \right]$

$\tilde{P}_\mu = P_\mu - \int d^3x\, F_0{}^\alpha \partial_\alpha A_\mu = P_\mu - \int d^3x\, F_0{}^i \partial_i A_\mu$

can restrict to $\alpha = i$ as F is antisymm.

But EOM says $\partial_i F^{i0} = 0$ as $\partial_\mu F^{\mu\nu} = 0$.

So $\tilde{P}_\mu = P_\mu - \int d^3x\, \partial_i (F_0{}^i A_\mu)$

$\to 0$ by choice of suitable BCs.

So the cons charges coming from $\tilde{T}$ & T are the same

<u>Next look at</u> <u>Lorentz tr</u> $x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu$

$\Rightarrow x^\mu = x'^\mu - \omega^\mu{}_\nu x'^\nu \Rightarrow \frac{\partial x^\mu}{\partial x'^\alpha} = \delta^\mu_\alpha - \omega^\mu{}_\alpha$

Now $A'_\mu(x^\alpha + \omega^\alpha{}_\beta x^\beta) = (\delta^\alpha_\mu - \omega^\alpha{}_\mu) A_\alpha(x) = A'_\mu(x) + \omega^{\alpha\beta} x_\beta \partial_\alpha A'_\mu(x)$

$= A_\mu(x) - \omega^\alpha{}_\mu A_\alpha(x)$

$\Rightarrow \boxed{\delta A_\mu(x) = \omega^{\alpha\beta} x_\alpha \partial_\beta A_\mu(x) + \omega_\mu{}^\alpha A_\alpha(x).}$

So $\delta S = \int d^4x\, \partial_\mu \left[\delta x^\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \delta A_\nu \right]$

$= \int d^4x\, \partial_\mu \left[\omega^{\mu\nu} x_\nu \mathcal{L} - F^{\mu\nu} (\omega^{\alpha\beta} x_\alpha \partial_\beta A_\nu + \omega_\nu{}^\alpha A_\alpha) \right]$

$= \int d^4x\, \omega^{\alpha\beta} \partial_\mu \left[\delta^\mu_\alpha x_\beta \mathcal{L} - F^\mu{}_\nu (x_\alpha \partial_\beta A_\nu - \eta_{\nu\beta} A_\alpha) \right] \to M_\mu{}^{\alpha\beta} \to$ 6 cons curr.

$M^\mu{}_{\alpha\beta} = \frac{1}{2} \left[(\delta^\mu_\alpha x_\beta - \delta^\mu_\beta x_\alpha) \mathcal{L} - F^{\mu\nu} (x_\alpha \partial_\beta - x_\beta \partial_\alpha) A_\nu + F^\mu{}_\nu (\delta^\nu_\beta A_\alpha - \delta^\nu_\alpha A_\beta) \right]$

(antisymm in $\alpha\beta$)

$$M^{\mu}{}_{\alpha\beta} = \frac{1}{2}\left[-x_\alpha\left(\delta^\mu{}_\beta \mathcal{L} + F^{\mu\nu}\partial_\beta A_\nu\right) + x_\beta\left(\partial^\mu{}_\alpha \mathcal{L} + F_\mu{}^\nu \partial_\alpha A_\nu\right)\right.$$ (7)

$$\left. + F^\mu{}_\beta A_\alpha - F^\mu{}_\alpha A_\beta\right]$$

Now do Belinfante. add & subtract. want it to look like $xT - xT$.

$$= \frac{1}{2}\left[-x_\alpha\left(\delta^\mu{}_\beta \mathcal{L} + F_\mu{}^\nu F_{\beta\nu} + F_\mu{}^\nu \partial_\nu A_\beta\right)\right.$$

$$+ x_\beta\left(\delta^\mu{}_\alpha \mathcal{L} + F_\mu{}^\nu F_{\alpha\nu} + F_\mu{}^\nu \partial_\nu A_\alpha\right)$$

$$\left. + F^\mu{}_\beta A_\alpha - F^\mu{}_\alpha A_\beta\right]$$

$$= \frac{1}{2}\Big[-x_\alpha \overbrace{\left(\delta^\mu{}_\beta \mathcal{L} + F_\mu{}^\nu F_{\beta\nu}\right)}^{\tilde{T}^\mu{}_\beta} + x_\beta \overbrace{\left(\delta^\mu{}_\alpha \mathcal{L} + F_\mu{}^\nu F_{\alpha\nu}\right)}^{\tilde{T}^\mu{}_\alpha}$$

$$\left. - x_\alpha F_\mu{}^\nu \partial_\nu A_\beta + x_\beta F_\mu{}^\nu \partial_\nu A_\alpha + F^\mu{}_\beta A_\alpha - F^\mu{}_\alpha A_\beta\right]$$

Consider last 4 term. Consider

$$-F_\mu{}^\nu \partial_\nu (x_\alpha A_\beta) + F_\mu{}^\nu \partial_\nu (x_\beta A_\alpha)$$

$$= -F_{\mu\alpha} A_\beta - F_\mu{}^\nu x_\alpha (\partial_\nu A_\beta) + F_{\mu\beta} A_\alpha + F_\mu{}^\nu x_\beta (\partial_\nu A_\alpha)$$

This matches the last 4 terms.

So $$\boxed{M^\mu{}_{\alpha\beta} = \underbrace{\frac{1}{2}\left[-x_\alpha \tilde{T}^\mu{}_\beta + x_\beta \tilde{T}^\mu{}_\alpha\right]}_{\tilde{M}^\mu{}_{\alpha\beta} \to \text{Belinfante current.}} + \frac{1}{2} F_\mu{}^\nu \partial_\nu (x_\beta A_\alpha - x_\alpha A_\beta)}$$

Must show that M & $\tilde{M}$ give same conserved quantites

$$\tilde{M}^\mu{}_{\alpha\beta} = M^\mu{}_{\alpha\beta} - \frac{1}{2} F^\mu{}_\nu \partial^\nu (x_\beta A_\alpha - x_\alpha A_\beta)$$

Cons chg $$\tilde{J}_{\alpha\beta} = \int d^3x\, \tilde{M}^0{}_{\alpha\beta} = \int d^3x \Big[M^0{}_{\alpha\beta} - \underbrace{\frac{1}{2} F^0{}_i \partial^i (x_\beta A_\alpha - x_\alpha A_\beta)}\Big]$$

by com $\partial_\nu[F^0{}_i(x_\beta A_\alpha - x_\alpha A_\beta)]$ $= 0$ as it is a Boundary term.

From now on we'll drop the tilde and use the Belinfante tensors

$$P_0 = \int d^3x\, T_{00} = \int d^3x \Big[-\frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} + \underbrace{F_0{}^\mu F_{0\mu}}_{F_0{}^i F_{0i}}\Big]$$

$$F_0{}^i F_{0i} = -F_{0i} F_{0i} = -\vec{E}^2.$$

$$F^{\alpha\beta} F_{\alpha\beta} = 2F^{0i} F_{0i} + F^{ij} F_{ij} = -2\vec{E}^2 + \underbrace{\epsilon_{ijk} B_k \epsilon_{ijl} B_l}_{2B^2}$$

$$= -2(E^2 - B^2) = -4\mathcal{L}$$

Thus $P_0 = \int d^3\bar{x}\left[\frac{1}{2}(E^2-B^2) - E^2\right] = -\frac{1}{2}\int d^3x\,(E^2+B^2) = -\int d^3x\,\mathcal{E}.$ 8

$$\boxed{P_0 = -\int d^3x\,\mathcal{E}}$$

Momentum of field: $P_i = \int d^3x\, T_{0i} = \int d^3x\, \underbrace{F_0{}^{j} F_{ij}} = -\int d^3x\, \underbrace{F_{0j} F_{ij}}_{E_j\,\epsilon_{ijk} B_k}$.

$$\boxed{\vec{P} = -\int d^3x\,(\bar{E}\times\bar{B}) = -\int d^3x\,(\text{poynting vector})}.$$

Ang mom: $J_{ij} = \int d^3x\, M^0{}_{ij} = \frac{1}{2}\int d^3x\left[x_j T_{0i} - x_i T_{0j}\right]$ use T_{0i}

$J_{ij} = \frac{1}{2}\int d^3x\left[-x_j(E\times\bar{B})_i + x_i(\bar{E}\times\bar{B})_j\right]$

$J_{ij} = \frac{1}{2}\int d^3x\left[x_i (E\times B)_j - x_j (E\times B)_i\right]$

$J_i = \frac{1}{2}\epsilon_{ijk} J_k = \frac{1}{2}\int d^3x\, \epsilon_{ijk}\, x_j (E\times B)_k.$

$$\boxed{\vec{J} = \frac{1}{2}\int d^3x\; \vec{r}\times(\bar{E}\times\bar{B})}$$

Note: Can directly get symmetrized stress tensor by varying action of fields coupled to fixed background metric.

References

1. Field Theory: A modern Primer (Pierre Ramond)
2. An Intro to QFT (Peskin & Schroeder)
3. Relativistic Quantum Fields (J D Borken & S. Drell). (Vol II)
3. Relativistic Quantum Fields (C. Itzykson & JB Zuber)
4. Quantum Field Theory

Next QFT of scalar field: Canonical Quantization.

$\varphi(x^\mu)$, $\Pi(x^\mu)$

QM $\to$ QFT
$p(t) \to \Pi(x)$
$x(t) \to \phi(x)$.

Equal time commutation relations

$[\varphi(\bar{x},t), \varphi(\bar{y},t)] = 0$

$[\Pi(\bar{x},t), \Pi(\bar{y},t)] = 0$

$[\varphi(x,t), \Pi(y,t)] = i\,\delta^3(\bar{x}-\bar{y})$

$\Pi = \dfrac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}$ for scalar

at unequal times get additional terms involving der of δ fn
Schwinger term

$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 \Rightarrow (\partial^\mu\partial_\mu + m^2)\phi = 0.$ do Mode expansion.

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[a_{\vec{p}}\, e^{-ip\cdot x} + a^{+}_{\vec{p}}\, e^{ip\cdot x}\right]\Big|_{p^0 = E_p = \sqrt{|\vec{p}|^2+m^2}}$$

check if it sats EOM.

$$(\Box + m^2)\phi = \int\frac{[d^3p]}{\sqrt{2E_p}}\left[-p^2+m^2\right]\left[a_p e^{-ip\cdot x} + a^{+}_p e^{ip\cdot x}\right] = 0$$

as $(p^0)^2 = |\vec{p}|^2 + m^2$

∞ # of SHOs. "Second quantization."

Comm relns among a, $a^\dagger$: $[a_{\vec p}, a_{\vec q}] = 0$

$[a^\dagger_{\vec p}, a^\dagger_{\vec q}] = 0$, $[a_{\vec p}, a^\dagger_{\vec q}] = (2\pi)^3 \delta^3(\vec p - \vec q)$

Note $a_p e^{-ip\cdot x} + a^\dagger_{\vec p} e^{ip\cdot x} = a_p(+\text{ve freq}) + a^\dagger_{\vec p}(-\text{ve freq})$

$e^{-ip\cdot x} \to e^{-i(\omega t - \vec k\cdot\vec x)} \to e^{i(k\cdot x - \omega t)} \to$ +ve freq.

$e^{ip\cdot x} \to e^{i(\omega t - \vec k\cdot \vec x)} \to e^{-i(k\cdot x - \omega t)} \to$ −ve freq.

$$\pi(x) = \int [d^3p] \frac{-iE_p}{\sqrt{2E_p}} \left(a_{\vec p} e^{-ip\cdot x} - a^\dagger_p e^{ip\cdot x} \right).$$

Next lecture: quantization of KG field, propagator. (Wed)
Thu & Fri → Dirac fld & its quantisation.
Fri & Sat → Quantization of Maxwell field.

2nd week: Wick expansion, S matrix, cross-section.
QED calc 1 cross section say $e^+e^- \to \mu^+\mu^-$.
also do Abelian Higgs Model & SSB.
last 2 lectures: Non-abelian gauge theory YM Lagrangian.

3rd lecture

$$H = \int d^3\vec x\, T_{00} = \int d^3x \left[\tfrac12 \pi^2 + \tfrac12 |\nabla\phi|^2 + \tfrac12 m^2\phi^2 \right].$$

$$H = \tfrac12 \int [d^3p]\, E_p \left(a^\dagger_p a_p + a_p a^\dagger_p \right) = \int [d^3p]\, E_p \left(a^\dagger_p a_p + \tfrac12 (2\pi)^3 \delta^3(\vec 0) \right)$$

drop this infinite energy by normal ordering.

$$:H: = \int \frac{d^3p}{(2\pi)^3} E_p a^\dagger_p a_p.$$ Henceforth drop the :: symbol

$$[H, a_p] = \int [d^3q] E_q [a^\dagger_q a_q, a_p] = \int [d^3q]\, -(2\pi)^3 \delta^3(p-q)\, a_q E_q$$

$[H, a_p] = -E_p a_p$ and $[H, a^\dagger_p] = E_p a^\dagger_p$

Suppose $|\phi\rangle$ is an egy e-state $H|\phi\rangle = E_0 |\phi\rangle$
then $a^\dagger_p|\phi\rangle$ ~~has~~ is also an egy e-state w/ egy $E_0 + E_p$.

Vacuum: $a_p|0\rangle = |0\rangle$ $\forall \vec p$ build Fock space.
$a_p|0\rangle = 0$ but get N particle state $a^\dagger_{p_N} \cdots a^\dagger_{p_2} a^\dagger_{p_1} |0\rangle$

Energy of this state is $E_n = E_{p_1} + E_{p_2} + \cdots + E_{p_n}$.

Similarly calc $\quad P^i = \int d^3x\, T^{0i} = \int d^3x\, \pi\, \partial^i \phi = -\int d^3x\, \pi\, \partial_i \phi$ (10)

$\vec{P} = -\int d^3x\, \pi\, \nabla\phi$ put in mode expansion to get

$\vec{P} = \int \frac{d^3q}{(2\pi)^3}\, \vec{q}\, \left(a_q^\dagger a_q + \frac{1}{2}(2\pi)^3 \delta^3(\vec{0}) \right)$ again ∞. So normal order: $:\vec{P}: = \int \frac{d^3q}{(2\pi)^3}\, \vec{q}\, a_q^\dagger a_q$. $\quad :\vec{P}: |0\rangle = 0$.

acting on 1 particle state $\quad \vec{P}\, a_q^\dagger |0\rangle = \vec{q}\, a_q^\dagger |0\rangle$.

Normalization $\quad \langle 0|0\rangle = 1$ by defn.

$|p\rangle \equiv \sqrt{2E_p}\, a_p^\dagger |0\rangle$. Why! Calc $\langle p|q\rangle = \sqrt{2E_p}\sqrt{2E_q}\,\langle 0| a_q a_p^\dagger |0\rangle$.

So $\langle p|q\rangle = (2\pi)^3 \delta^3(p-q)\, 2E_p$.

E_p is rotn inv. but tr as time cpt of 4 vec. $\delta^3(p-q)$ is rotn inv

Will show this is inv under boosts.

$E_p\, \delta^3(\vec{p}-\vec{q})$. is inv under boosts

boost in $\hat{z}$ dir $\quad \begin{pmatrix} E' \\ p^{3'} \end{pmatrix} = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} E \\ p^3 \end{pmatrix} \qquad \cosh\eta = \frac{1}{\sqrt{1-v^2}}$.

$$\delta^3(\vec{p}'-\vec{q}') = \frac{\delta^3(\vec{p}-\vec{q})}{\left|\frac{dp_3'}{dp_3}\right|} = \frac{\delta^3(\vec{p}-\vec{q})}{\left|ch + sh\,\frac{dE}{dp_3}\right|} \qquad p^{3'} = ch\, p^3 + sh\, E$$

$$E = \sqrt{p_3^2 + m^2}$$

$$\frac{dE}{dp_3} = \frac{p_3}{E}.$$

$$= \frac{\delta^3(\vec{p}-\vec{q})}{\left(ch + sh\,\frac{p_3}{E}\right)} = \frac{E_p}{E_p'}\,\delta^3(p-q)$$

So $\quad E_p'\, \delta^3(\vec{p}'-\vec{q}') = E_p\, \delta^3(p-q)$ So this qty is inv under Lorentz boosts.

Can define projector $\quad (1)_{\substack{\text{single}\\ \text{particle}\\ \text{states}}} = \int \frac{d^3p}{(2\pi)^3}\, \frac{|p\rangle\langle p|}{2E_p}$.

check this is consistent.

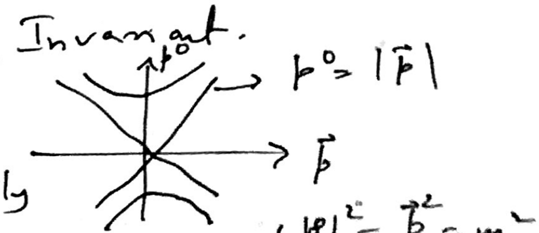
$$\langle \vec{r}|\vec{s}\rangle = \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{2E_p}\, \underbrace{\langle r|p\rangle}_{(2\pi)^3 2E_p \delta^3(r-p)}\, \langle p|\vec{s}\rangle \qquad \langle p|\vec{s}\rangle \to (2\pi)^3\, 2E_p\, \delta^3(p-s)$$

$$= (2\pi)^3\, 2E_r\, \delta^3(r-s) \quad \text{as desired}$$

Will show that $\int \frac{d^3p}{(2\pi)^3}\, \frac{1}{2E_p}$ is Lorentz Invariant.

Consider $\int \frac{d^4p}{(2\pi)^4}\, 2\pi\, \delta(p^2-m^2)\Big|_{p^0>0}$ → This is manifestly Lorentz invariant.

Claim these two expr are same. So the former is also inv under (proper ortho) LT.



$(p^0)^2 - \vec{p}^2 = m^2$ gives 2 sheeted hyperboloid.

proper orthochronous LT $\det \Lambda = 1,\ \Lambda^0{}_0 \geq 1$

Conn cpt of identity

do the integral $\int \frac{d^4p}{(2\pi)^4} 2\pi \delta(p^2 - m^2)\Big|_{p^0>0}$

$= \int \frac{d^3p}{(2\pi)^3} \int dp^0\, \delta(p^{0\,2} - (\vec{p}^{\,2} + m^2)) = \int \frac{d^3p}{(2\pi)^3} \int dp^0 \left[\frac{\delta(p^0 - E_p)}{2E_p} + \cancel{\frac{\delta(p^0 + E_p)}{2E_p}} \right]\Bigg|_{p^0>0}$

$= \int \frac{d^3p}{2\pi^3} \frac{1}{2E_p}$. as desired.

orthochronous			non orthochro.			
$\det \Lambda = 1$	$\Lambda^0{}_0 \geq 1$	$L^\uparrow_+$	$\det \Lambda = 1$	$\Lambda^0_0 \leq -1$	$L^\downarrow_+$	proper
$\det \Lambda = -1$	$\Lambda^0{}_0 \geq 1$	$L^\uparrow_-$	$\det \Lambda = -1$	$\Lambda^0_0 \leq -1$	$L^\downarrow_-$.	improp

Conn cpt of identity rotn & boosts. eg $\Lambda = \begin{pmatrix} 1 & \\ & \text{rotn} \end{pmatrix}$; $L^\downarrow_+ : \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$L^\uparrow_- : \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$; $L^\downarrow_- : \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

Propagator: $S = \int d^4x \left(-\frac{1}{2}\right) \phi \left(\partial_\mu \partial^\mu + m^2\right) \phi$ up to integ by parts.

$(\partial_\mu \partial^\mu_x + m^2) D(x-y) = -i\delta^4(x-y)$ work in mom space

$D(x-y) = \int \frac{d^4p}{(2\pi)^4} D(p)\, e^{-ip\cdot(x-y)} \Rightarrow \int \frac{d^4p}{(2\pi)^4} (-p^2 + m^2) D(p) e^{-ip\cdot(x-y)}$

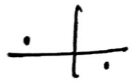
$= -i \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)}$

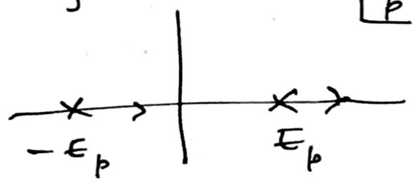
So $D(p)(-p^2 + m^2) = -i$

$D(p) = \frac{i}{p^2 - m^2}$ $\qquad \boxed{D(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)}}$

How to do integral? complex p^0 plane integ. has poles $\lfloor p^0$

push them off the real axis



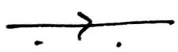


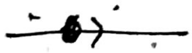
$e^{-ip^0(x^0 - y^0)} \to e^{\pm\epsilon(x^0 - y^0)}$

$\frac{1}{(p_0)^2 - \sqrt{p^2 + m^2} \pm i\epsilon}$ $\qquad \frac{1}{E - E_0 + i\Gamma}$.

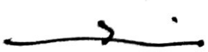
both particle & anti particle must prop forw in time.

Next: Feynman greens fn Dirac field


particle forw in time
anti back


particle back
anti forw


Time ordered both forw in time


anti time ordered both back in time

Review of ~~totes~~ Lorentz gp: $x^\mu x_\mu$ inv $= x'^\mu x'_\mu \Rightarrow \eta_{\mu\nu} = \eta_{\alpha\beta}\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu$

$\Lambda = \Lambda^\mu{}_\nu$ $\mu \to$ row $\nu =$ column. So $(\Lambda^T)_\mu{}^\alpha \eta_{\alpha\beta}\Lambda^\beta{}_\nu = \eta_{\mu\nu}$

treat $\eta_{\mu\nu}$ as a matrix. so $\Lambda^T\eta\Lambda=\eta$ so $(\det\Lambda)^2=1$.

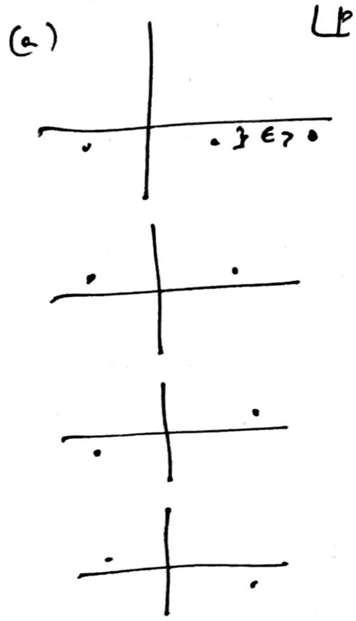
$\det\Lambda=\pm1$. Now $\eta_{00}=1=\eta_{\alpha\beta}\Lambda^\alpha{}_0\Lambda^\beta{}_0 = (\Lambda^0{}_0)^2-\sum_1^3(\Lambda^i{}_0)^2$.

$\Rightarrow (\Lambda^0{}_0)^2 = 1+\sum_1^3(\Lambda^i{}_0)^2 \geq 1$.

So we have $(\det\Lambda)^2=1$ and $|\Lambda^0{}_0|\geq 1$.

Propagator for KG: $D(x-y)=\int\frac{d^4p}{(2\pi)^4}\cdot\frac{i}{p^2-m^2}\cdot e^{-ip\cdot(x-y)}$.

(a)



move poles. denom $= p_0^2-(\vec p^{\,2}+m^2) = p_0^2-E_p^2$

$= (p_0-E_p)(p_0+E_p)$.

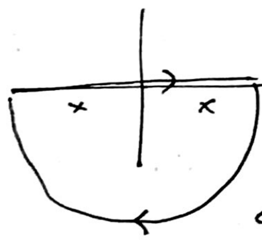
Case (a)

then poles are at $[p^0-(E_p-i\epsilon)][p^0-(-E_p-i\epsilon)]$.

$= ((p_0+i\epsilon)-E_p)((p_0+i\epsilon)+E_p)$

ϵ small, so expand

$=(p^0+i\epsilon)^2-E_p^2 = (p^0)^2+\underline{2i\epsilon p^0}-E_p^2$ extra term.



$$\int dp^0 \frac{e^{-ip^0(x-y)^0}}{[p^0-(E_p-i\epsilon)][p^0-(-E_p-i\epsilon)]}$$

clock wise contour so −ve sign. use Cauchy.

$$= -2\pi i\left[\frac{e^{-i(E_p-i\epsilon)(x-y)^0}}{(E_p-i\epsilon)-(-E_p-i\epsilon)}+\frac{e^{-i(-E_p-i\epsilon)(x-y)^0}}{(-E_p-i\epsilon)-(E_p-i\epsilon)}\right]$$

$$= -2\pi i\left[\frac{e^{-iE_p(x^0-y^0)-\epsilon(x-y)^0}}{2E_p}-\frac{e^{iE_p(x^0-y^0)-\epsilon(x-y)^0}}{2E_p}\right]$$

take x^0-y^0 large want it to die off, not blow up for this need $x^0>y^0$

$$= -2i\pi\,\theta(x^0-y^0)\left[\frac{e^{-iE_p(x-y)^0}}{2E_p}-\frac{e^{iE_p(x-y)^0}}{2E_p}\right] \quad (*)$$

This is retarded Greens fn need $x^0>y^0$ non zero.

On other hand, if you take both poles to upper ½ plane sign of ϵ is reversed & you get $\theta(y^0-x^0)$ get advanced green fn.

To understand antiparticles, need to look at complex KG eqn

$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi$ $\quad \phi=\frac{\phi_1+i\phi_2}{\sqrt2}$ $\quad \phi^*=\frac{\phi_1-i\phi_2}{\sqrt2}$.

Mode exp. $\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a_p e^{-ip\cdot x} + b_p^+ e^{ip\cdot x} \right] \Big|_{p^0 = E_p}$

$H = \int [d^3p] \, E_p \, (a_p^+ a_p + b_p^+ b_p)$.

$\phi(x) \to e^{i\alpha} \phi(x)$ invariance cons curr

$J^\mu = i \{ \phi^* \partial^\mu \phi - \phi \partial^\mu \phi^* \}$ $\quad Q = \int d^3x \, J^0 = \int [d^3p] (a_p^+ a_p - b_p^+ b_p)$

1-particle state $a_p^+ |0\rangle$ $\quad$ 1 antiparticle $b_p^+ |0\rangle$.

have same mass (egy) but opposite charge Q.

here if you calc $\langle 0 | [\phi(x), \phi^+(y)] | 0 \rangle$ → get same as (*) on last page

$\langle 0 | \underset{a}{\phi(x)} \, \underset{a^+}{\phi^+(y)} | 0 \rangle \to$ $\quad y^0 \xrightarrow{\text{particle}} x^0 > y^0$. particle moving forw

→ time

similarly $\langle 0 | \underset{b_p}{\phi^+(y)} \, \underset{b_p^+}{\phi(x)} | 0 \rangle$ $\quad y^0 \xleftarrow{\text{antiparticle}} x^0 > y^0$. antipar mov back in time

So this retarded Green fn is not acceptable as it describes antiparticles moving back in time

Similarly Advanced green fn $\langle 0 | [\phi(y), \phi^+(x)] | 0 \rangle$. will describe particle moving back in time & antiparticle moving forw in time.

So we have ruled out and

Now back to real scalar, try $\qquad -E_p + i\epsilon \qquad E_p - i\epsilon \qquad \epsilon > 0$

denom is $[p^0 - (E_p - i\epsilon)][p^0 - (-E_p + i\epsilon)]$

$= [p^0 - (E_p - i\epsilon)][p^0 + (E_p - i\epsilon)]$

$= (p^0)^2 - (E_p - i\epsilon)^2 = (p^0)^2 - E_p^2 + \underbrace{i\epsilon}_{2i\epsilon E_p = i\epsilon'} = p^2 - m^2 + i\epsilon$

So $D_F(x-y) = \int [d^4p] \frac{\exp(-ip(x-y))}{p^2 - m^2 + i\epsilon}$ $\quad$ "$i\epsilon$" prescription. Feynman Green fn.

look at p^0 integ.

$= \int dp^0 \frac{e^{-ip^0(x^0 - y^0)}}{[p^0 - (E_p - i\epsilon)][p^0 - (-E_p + i\epsilon)]}$

here omit the space integral

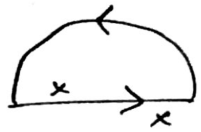
$e^{-\epsilon(x-y)^0} \Rightarrow \theta(x^0 - y^0)$

$e^{-i(E_p - i\epsilon)(x^0 - y^0)}$

for 1st pole (left factor)

for $\dfrac{1}{p^0-(-E_p+i\epsilon)}$ use $e^{-i(-E_p+i\epsilon)(x^0-y^0)} \sim e^{\epsilon(x-y)^0}$

$\Rightarrow \theta(y^0-x^0)$

So must close up  damping when $y^0>x^0$.

$$D(x-y) = -2\pi i \frac{e^{-iE_p(x^0-y^0)}}{\underbrace{(E_p-i\epsilon)-(-E_p+i\epsilon)}_{2E_p \text{ as } \epsilon\to 0^+}}\theta(x^0-y^0) + 2\pi i \frac{e^{iE_p(x-y)^0}}{\underbrace{(-E_p+i\epsilon)-(E_p-i\epsilon)}_{-2E_p}}\theta(y^0-x^0)$$

$$D(x-y) = -2\pi i\,\theta(x^0-y^0)\frac{e^{-iE_p(x-y)^0}}{2E_p} - 2\pi i\,\theta(y^0-x^0)\frac{e^{iE_p(x-y)^0}}{2E_p}$$

So $D_F(x-y) = \theta(x^0-y^0)\int\frac{[d^3p]}{2E_p}\, e^{-iE_p(x-y)^0+i\vec{p}\cdot(\vec{x}-\vec{y})} \to e^{-ip(x-y)}\Big|_{p^0=E_p}$

$+\,\theta(y^0-x^0)\int\frac{[d^3p]}{2E_p}\, e^{iE_p(x-y)^0-i\vec{p}\cdot(\vec{x}-\vec{y})} \to e^{ip\cdot(x-y)}\Big|_{p^0=E_p}$

So Feynman Green fn $\nearrow$ can write in terms of VEVs.

Now $\langle 0|\phi(x)\phi(y)|0\rangle = \langle 0|\int\frac{[d^3q]}{\sqrt{2E_q}}\, a_q\, e^{-iq\cdot x}\int\frac{[d^3p]}{\sqrt{2E_p}}\, a^\dagger_p\, e^{ip\cdot y}|0\rangle$.

Notice other terms dont contrib

$$= \int\frac{d^3p}{(2\pi)^3}\frac{e^{-ip\cdot(x-y)}}{2E_p}$$

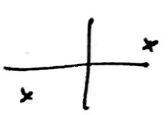
So $D_F(x-y) = \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle$.

$\equiv \langle 0|T\,\phi(x)\,\phi(y)|0\rangle$. Time ordered product

Thus $D_F(x-y) = \int[d^4p]\dfrac{e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}\Big|_{\epsilon\to 0^+} = \langle 0|T\,\phi(x)\phi(y)|0\rangle$.

Complex KG case $D_F(x-y) = \theta(x^0-y^0)\langle 0|\phi(x)\,\phi^\dagger(y)|0\rangle$

$+\,\theta(y^0-x^0)\langle 0|\phi^\dagger(y)\,\phi(x)|0\rangle$.

 leads to antitime ordered both particle & antiparticle move back in time.

$D_F(x-y) = \int[d^4p]\, e^{-ip\cdot(x-y)}\,\tilde{D}_F(p)$ where $D_F(p) = \dfrac{i}{p^2-m^2+i\epsilon}$

retarded green fn used in EM.

Feynman diagram for scalar propagator $\xrightarrow{\quad}{}_{p} \;=\; \dfrac{i}{p^2-m^2+i\epsilon}$

Next Classical Dirac field (Quant. of Dirac tomorrow)

Dirac field under LT tr as $\psi(x) \to e^{-\frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}}\psi$

where $S_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$. $\quad \psi \to$ 4 cpt col. spinor.

$$S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] \quad \text{gen of boost.}$$

$$= \frac{i}{4}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

$$= \frac{i}{4}\left[\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} - \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}\right]$$

$$= -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}.$$

$$\sigma^\mu = (\mathbb{1}, \vec{\sigma})$$

$$\bar{\sigma}^\mu = (\mathbb{1}\ -\vec{\sigma}).$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_{4\times 4} \quad \text{Clifford,}$$

$\omega^{0i} = \beta^i = $ Boost params ~~(angles)~~ v^i cosh(ang)

$\omega^{ij} = \epsilon^{ijk}\theta^k = $ Rotn angles

$S^{0i} \to$ spinor rep of boosts

$S^{ij} \to$ " " " rotns.

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{i}{4}\left[\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} - (i \leftrightarrow j)\right]$$

$$= \frac{i}{4}\left[\begin{pmatrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{pmatrix} - (i \leftrightarrow j)\right] \qquad [\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k.$$

$$= -\frac{i}{4}\begin{pmatrix} [\sigma^i, \sigma^j] & 0 \\ 0 & [\sigma^i, \sigma^j] \end{pmatrix} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

$$\psi \to e^{-i\omega^{0i}S_{0i} - i\omega^{ij}S_{ij}} = e^{-\frac{1}{2}\underbrace{\omega^{0i}}_{v^i}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} - \frac{i}{2}\epsilon_{ijk}\omega_{ij}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}}$$

$$\psi \to e^{-\frac{1}{2}\vec{v}\cdot\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix} + \frac{i}{2}\vec{\omega}\cdot\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}}\psi$$

$$\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \to \begin{pmatrix} e^{\frac{i\vec{\sigma}}{2}\cdot(\vec{\omega} + i\vec{v})} & 0 \\ 0 & e^{\frac{i\vec{\sigma}}{2}(\vec{\omega} - i\vec{v})} \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \quad \text{block digonal}$$

so the rep of Lorentz is reducible into the L & R subspaces

So can write $\psi_L \to \Lambda_L\psi_L \qquad \Lambda_L = \exp\left(\frac{i}{2}\vec{\sigma}\cdot(\vec{\omega} + i\vec{v})\right)$

$\psi_R \to \Lambda_R\psi_R \qquad \Lambda_R = \exp\left(\frac{i}{2}\vec{\sigma}\cdot(\vec{\omega} - i\vec{v})\right).$

No RH neutrinos in SM.

SO(10) GUT has RH neutrino.

ν_R ν_L e_L e_R u_L d_L u_R d_R. 1+15 dim 16 dim spinor rep of SO(10)

$\nu_R \downarrow$ singlet under all SM gauge fields; $u_L \downarrow 3$, $d_L \downarrow 3$, $u_R \downarrow 3$, $d_R \downarrow 3$ (colors)

Dirac eqn $(i\not\partial - m)\psi = 0$ implies $\bar\psi(i\not\partial)\psi = 0$

for m=0. $\psi^\dagger \gamma^0 i\gamma^\mu \partial_\mu \psi = (\psi_L^\dagger\ \psi_R^\dagger)\begin{pmatrix}0&1\\1&0\end{pmatrix} i \begin{pmatrix}0&\sigma^\mu\\ \bar\sigma^\mu & 0\end{pmatrix}\partial_\mu\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}$

$= (\psi_L^\dagger\ \psi_R^\dagger)\, i\begin{pmatrix}\bar\sigma^\mu & 0\\ 0 & \sigma^\mu\end{pmatrix}\partial_\mu\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = i\psi_L^\dagger\bar\sigma^\mu\partial_\mu\psi_L + i\psi_R^\dagger\sigma^\mu\partial_\mu\psi_R.$

no cross term.

$\psi_L^\dagger\bar\sigma^\mu\partial_\mu\psi_L = 0$ Dirac eqn for LH spinors (m=0)

$\psi_R^\dagger\sigma^\mu\partial_\mu\psi_R = 0$ " " " RH " (m=0)

Can build up Dirac eqn from these two.

ψ_L, χ_L $\chi_L^T\sigma_2\psi_L \to \chi_L^T\underbrace{(\Lambda_L^T\sigma_2\Lambda_L)}_{\to\sigma_2 i\frac{\vec\sigma}{2}(\vec\omega+i\vec v)}\psi_L.$

$\sigma_2\Lambda_L^T\sigma_2\Lambda_L = \sigma_2 e^{i\frac{\vec\sigma}{2}^T(\vec\omega+i\vec v)}\sigma_2\, e^{i\frac{\vec\sigma}{2}(\vec\omega+i\vec v)} = \mathbb{1}$

$(\sigma_2\vec\sigma^T\sigma_2 = -\vec\sigma)$ $= e^{-\frac{i}{2}\vec\sigma(\vec\omega+i\vec v)}\, e^{i\frac{\vec\sigma}{2}(\vec\omega+i\vec v)}$

This gives a way of giving mass to LH spinors alone

this is Majorana mass.

$\psi_L^T\sigma_2\psi_L = \underline{-i(\psi_1\psi_2 - \psi_2\psi_1)} = (\psi_1\ \psi_2)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$

$\mathcal{L} = \bar\psi i\not\partial\psi$ $\psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}$ $\sigma^\mu = (1,\vec\sigma)$, $\bar\sigma^\mu = (1,-\vec\sigma)$

$= i\psi_L^\dagger\bar\sigma^\mu\partial_\mu\psi_L + i\psi_R^\dagger\sigma^\mu\partial_\mu\psi_R.$

$\psi_L \to \Lambda_L\psi_L$, $\Lambda_L = \exp\left(\frac{i\vec\sigma}{2}\cdot(\vec\omega+i\vec v)\right)$

$\psi_R \to \Lambda_R\psi_R$ $\Lambda_R = \exp\left(i\frac{\vec\sigma}{2}\cdot(\vec\omega - i\vec v)\right)$ } Weyl Fermions.

$\gamma^5 = \begin{pmatrix}-1&0\\0&1\end{pmatrix}$ $(\gamma^5)^2 = 1.$ $P_\pm = \frac{1}{2}(1\pm\gamma^5)$ Projn ops.

$P_\pm^2 = P_\pm$ $P_-\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = \begin{pmatrix}\psi_L\\ 0\end{pmatrix}$ $P_+\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = \begin{pmatrix}0\\ \psi_R\end{pmatrix}$

ψ_L, χ_L $\psi_L^T\sigma_2\chi_L$: Lorentz scalar.

$\Lambda_L^T\sigma_2\Lambda_L = \sigma_2$ $(\psi_1\psi_2)^\dagger = \psi_2^\dagger\psi_1^\dagger$

Want some real #s. $\psi_L^T \sigma_2 \psi_L$ is a Lorentz scalar but not real (17)

$m(\psi_L^T \sigma_2 \psi_L + \psi_L^\dagger \sigma_2 \psi_L^*) = \mathcal{L}_m^L = z + \bar{z}$ | take Re and Im

also have $z - \bar{z}$.

$im(\psi_L^T \sigma_2 \psi_L - \psi_L^\dagger \sigma_2 \psi_L^*) = \mathcal{L}_m^{L,5} = z - \bar{z}$

$\sigma_2 \psi_L^*$ under LT $\longrightarrow \sigma_2 \Lambda_L^* \psi_L^* = \sigma_2 \Lambda_L^* \sigma_2 \sigma_2 \psi_L^*$

as $\sigma_2^2 = I$. Now $\sigma_2 \Lambda_L^* \sigma_2 = \sigma_2 \exp(-i\frac{\vec{\sigma}^*}{2}(w - iv))\, \sigma_2$.

check that $\sigma_2 \vec{\sigma} \sigma_2 = -\vec{\sigma}^*$ or $\sigma_2 \vec{\sigma}^* \sigma_2 = -\vec{\sigma}$

So becomes $\sigma_2 \Lambda_L^* \sigma_2 = \exp(i \frac{\vec{\sigma}}{2} \cdot (w - iv)) = \Lambda_R$.

So $\psi_L \to$ left mover. $\sigma_2 \psi_L^*$ is a right mover.

Define charge conj

$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \qquad \psi^c = \begin{pmatrix} \sigma_2 \psi_R^* \\ -\sigma_2 \psi_L^* \end{pmatrix}$. check that $(\psi^c)^c = \psi$.

$(\psi^c)^c = \begin{pmatrix} \sigma_2 \psi_R^* \\ -\sigma_2 \psi_L^* \end{pmatrix}^c = \begin{pmatrix} \sigma_2 (-\sigma_2 \psi_L^*)^* \\ -\sigma_2 (\sigma_2 \psi_R^*) \end{pmatrix} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \psi$.

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & - \end{pmatrix}$ $\qquad \sigma_2^* = -\sigma_2 \qquad (-\sigma_2)^T = \sigma_2$

Majorana spinor: $\psi^M = \begin{pmatrix} \psi_L \\ -\sigma_2 \psi_L^* \end{pmatrix}$. has same dof as a single Weyl spinor

convention. to go w/ defn of chg conjugation.

What is $(\psi^M)^c = \begin{pmatrix} \sigma_2 (-\sigma_2 \psi_L^*)^* \\ -\sigma_2 \psi_L^* \end{pmatrix} = \psi^M$.

Now re-write ~~Majorana~~ Weyl theory in terms of Majorana fermions

Many believe neutrinos are Majorana spinors.

(Dirac theory has a redundancy (reducibility) if Dirac mass $= 0$).

$\mathcal{L} = \frac{i}{2} \bar{\psi}_M \gamma^\mu \partial_\mu \psi_M = \frac{i}{2} \psi_M^\dagger \gamma^0 \gamma^\mu \partial_\mu \psi_M$

$= \frac{i}{2} (\psi_L^\dagger \quad -\psi_L^T \sigma_2) \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}}_{\begin{pmatrix} \bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix}} \partial_\mu \begin{pmatrix} \psi_L \\ -\sigma_2 \psi_L^* \end{pmatrix}$.

$= \frac{i}{2} [\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + \psi_L^T \sigma_2 \sigma^\mu \sigma_2 \partial_\mu \psi_L^*]$

check

Now use identity $\sigma^2 \sigma^\mu \sigma^2 = \bar{\sigma}^{\mu T}$: if $\mu = 0$ $(\sigma^2)^2 = I$. etc

$\mathcal{L} = \frac{i}{2} [\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + \psi_L^T \bar{\sigma}_\mu^T \partial_\mu \psi_L^*]$ $\qquad \psi_{L\alpha} \bar{\sigma}^\mu_{\beta\alpha} \partial_\mu \psi_{L\beta}^* = -\partial_\mu \psi_{L\beta}^* \bar{\sigma}^\mu_{\beta\alpha} \psi_{L\alpha}$

$= -\partial_\mu \psi_L^\dagger \bar{\sigma}^\mu \psi_L \to \psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L$

Majorana Lagrangian (Friday 24 Aug Anirban)

Thus $\mathcal{L} = \frac{i}{2} \bar{\Psi}_M \gamma^\mu \partial_\mu \Psi_M = \mathcal{L}^{weyl} = i \psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L$.

Now want to add a mass term. to this majorana $\mathcal{L}$

Try $\mathcal{L}_M^L = -\frac{m}{2} \bar{\Psi}_M \Psi_M = -\frac{m}{2} \Psi_M^\dagger \gamma^0 \Psi_M$

$\mathcal{L}_M^L = -\frac{m}{2} \left(\psi_L^\dagger \;\; -\psi_L^T \sigma_2 \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ -\sigma_2 \psi_L^* \end{pmatrix}$

$= \frac{m}{2} \left(\psi_L^\dagger \sigma_2 \psi_L^* + \psi_L^T \sigma_2 \psi_L \right) \rightarrow$ This is same as the mass term introduced before

So this is the Majorana mass term.

other possibility is a pseudo scalar

$\mathcal{L}_M^{L,5} = -i\frac{m}{2} \bar{\Psi}_M \gamma^5 \Psi = -\frac{im}{2} \left(\psi_L^\dagger \;\; -\psi_L^T \sigma_2 \right) \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \begin{pmatrix} \psi_L \\ -\sigma_2 \psi_L^* \end{pmatrix}$

$= \frac{im}{2} \left(\psi_L \sigma_2 \psi_L^* - \psi_L^T \sigma_2 \psi_L \right) \rightarrow$ this agrees w/ earlier mass term in the Weyl theory. pseudo scalar parity violating.

So we can have massive purely Left handed fermions.

Now want to get a Dirac mass term. by combining L & R.

$\chi_L = \sigma_2 \psi_R^*$ is a Left handed. $\psi_R^\dagger \psi_L$ is a scalar

$-\chi_L^T \sigma_2 \psi_L$ is Lorentz scala. So can construct a mass term

Consider $\mathcal{L}_D = m(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R)$

or $\mathcal{L}_D^5 = im \left(\psi_R^\dagger \psi_L - \psi_L^\dagger \psi_R \right)$.

Compare w/ $m\bar{\Psi}\Psi = m \left(\psi_L^\dagger \;\; \psi_R^\dagger \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

$m\bar{\Psi}\Psi = m \left(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L \right) = \mathcal{L}_D$.

Try also $im \bar{\Psi} \gamma^5 \Psi = im \left(\psi_L^\dagger \;\; \psi_R^\dagger \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$.

$im \bar{\Psi} \gamma^5 \Psi = im \left(\psi_L^\dagger \;\; \psi_R \right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = im \left(\psi_L^\dagger \psi_R - \psi_R^\dagger \psi_L \right)$.

this is pseudo scalar Dirac mass term

Dirac mass couples LH & RH fermions. Kinetic term does not couple them.

Internal Symm

$\psi_L \rightarrow e^{i\alpha_L} \psi_L$, $\psi_R \rightarrow e^{i\alpha_R} \psi_R$ $\quad \alpha_L, \alpha_R$ const.

Now $\mathcal{L}_L = i \psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L$ & $\mathcal{L}_R = i \psi_R^\dagger \sigma^\mu \partial_\mu \psi_R$ are both inv.

Further if $\alpha_L = \alpha_R = \alpha$, then $\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \rightarrow \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = e^{i\alpha} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

$\Psi(x) \rightarrow e^{i\alpha} \Psi(x)$ Vector transformation of Dirac field. symm of $\mathcal{L} = \bar{\Psi} i \partial\!\!\!/ \Psi$.

On other hand if take $\alpha_L = -\alpha_R = \beta$ then

$$\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} \to \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta}\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}.$$

This is still a symm of $\mathcal{L}_L + \mathcal{L}_R$.

Can we write more compactly?

Consider $e^{i\beta\gamma^5} = I + i\beta\gamma^5 - \frac{\beta^2 \overbrace{(\gamma^5)^2}^{I}}{2!} - i\frac{\beta^3\gamma^5}{3!} + \dots$

$(\gamma^5)^2 = I$.

$\gamma^5 = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$

$$= I\left(1 - \frac{\beta^2}{2!} + \dots\right) + i\gamma^5\left(\beta - \frac{\beta^3}{3!} + \dots\right).$$

$$= I\cos\beta + i\sin\beta\,\gamma^5.$$

$$= \begin{pmatrix}\cos\beta - i\sin\beta & 0\\ 0 & \cos\beta + i\sin\beta\end{pmatrix} = \begin{pmatrix} e^{-i\beta} & 0\\ 0 & e^{i\beta}\end{pmatrix}$$

Thus by redefining $\beta \to -\beta$. we see that taking $\alpha_L = -\alpha_R$.
This is chiral or axial transformation. $\psi \to e^{-i\beta\gamma^5}\psi$.
This is a symm of Dirac kinetic term.
What about mass term in Dirac?

$m\bar\psi\psi$ is symm under $U(1)_V$.

$\bar\psi\psi = \psi^\dagger\gamma^0\psi \to \psi^\dagger e^{i\beta\gamma^5}\gamma^0 e^{-i\beta\gamma^5}\psi$.

Now $\gamma^5 = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{1}{4!}\epsilon_{\mu\nu\lambda\rho}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho$.

$\{\gamma^5, \gamma^\mu\} = 0$

So $\bar\psi\psi \to \psi^\dagger\gamma^0 e^{-2i\beta\gamma^5}\psi \neq \bar\psi\psi$.

So mass term violates $U(1)_{Axial}$. Axial curr is cons only if $m=0$. curr is a pseudo vector.

(Axial symm is broken in quantum theory even if $m=0$)
axial anomaly

$U(3)_L \times U(3)_R$

u_L	u_R
d_L	d_R
s_L	s_R

QCD vac. breaks to $U(3)_{L+R}$ diagonal

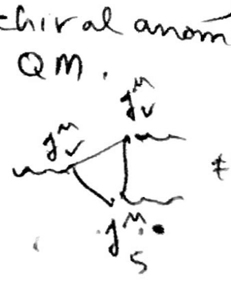
expect as many NG bosons as there are broken symm, but get 1 less. Expect 9 massless fermions but get only 8 → Gellmann's octet

$U(3)_L = U(1)_L \times SU(3)_L$ $\qquad$ $U(3)_R = U(1)_R \times SU(3)_R$

get $U(1)_{L+R} = U(1)_V \to$ Baryon #

$U(1)_{L-R} = U(1)_A \to$ broken by chiral anomal in QM.

$\underbrace{}_{\text{massless}}$ $\cdot\,\eta \to$ massive



't Hooft's resolution of η problem.

$U(N) = SU(N)\times U(1)/\text{center}$ $\qquad$ $e^{i\vec\sigma\cdot\vec\theta/2 + i\theta_0 I}$

Now construct conserved currents for $U(1)_V$ & $U(1)_A$ & check $U(1)_A$ cons only if $m=0$

$\mathcal{L} = i\bar\psi \not\partial \psi - m\bar\psi\psi$; $\psi(x) \to e^{i\alpha}\psi(x)$

$\delta\psi = i\alpha\psi$.
$\delta\bar\psi = -i\alpha\bar\psi$

$\delta S = \int d^4x\, \partial_\mu[\underbrace{\cancel{\delta x^\mu \mathcal{L}}}_{0} + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi]$

$\frac{\partial \mathcal{L}}{\partial \partial_\mu \bar\psi} = 0$

$\int \partial_\mu[\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi} i\alpha\psi] = -\int d^4x\, \alpha\partial_\mu(\bar\psi\gamma^\mu\psi)$

$\bar\psi\overleftarrow{\not\partial} = (\partial_\mu\bar\psi)\gamma^\mu$.

So $J^\mu_V = i\bar\psi\gamma^\mu\psi$

Now EOM is $(i\not\partial - m)\psi = 0$ for $\bar\psi$

$\delta S = \int d^4x\, \bar\psi(i\not\partial - m)\delta\psi = \int d^4x\, \bar\psi(-i\overleftarrow{\not\partial} - m)\delta\psi$

So $-i\partial_\mu\bar\psi\gamma^\mu - m\bar\psi = 0$ or $\boxed{\bar\psi(i\overleftarrow{\not\partial} + m) = 0}$

Now check cons of J^μ

$\partial_\mu J^\mu_V = i\partial_\mu\bar\psi\gamma^\mu\psi + i\bar\psi\gamma^\mu\partial_\mu\psi$.
$= i\bar\psi\overleftarrow{\not\partial}\psi + i\bar\psi\not\partial\psi = -m\bar\psi\psi + m\bar\psi\psi = 0$.

Similarly do Axial symm.

$J^\mu_5 = i\bar\psi\gamma^\mu\gamma^5\psi$ Calc $\partial_\mu J^\mu_5 = i\bar\psi\overleftarrow{\not\partial}\gamma^5\psi + i\bar\psi\underbrace{\gamma^\mu\gamma^5}_{-\gamma^5\gamma^\mu}\partial_\mu\psi$

$\partial_\mu J^\mu_5 = i\bar\psi\overleftarrow{\not\partial}\gamma^5\psi - i\bar\psi\gamma^5\not\partial\psi = -m\bar\psi\gamma^5\psi - m\bar\psi\gamma^5\psi$

$\neq 0$ in general vanishes only if $m=0$.

$\partial_\mu J^\mu_5 = -2m\bar\psi\gamma^5\psi$

Charge is $Q = \int d^3x\, \psi^\dagger\psi$. for vector $U(1)_V$.

$Q^5 = \int d^3x\, \psi^\dagger\gamma^5\psi$ for axial $U(1)_A$.

or $Q = \int d^3x(\psi_L^\dagger\psi_L + \psi_R^\dagger\psi_R)$ & $Q^5 = \int d^3x(-\psi_L^\dagger\psi_L + \psi_R^\dagger\psi_R)$.

Next Quantize Dirac Theory

Need plane wave solns of Dirac eqn (classical) → spinors then use them to get Mode expansion.

$(i\not\partial - m)\psi = 0$ mult by $(i\not\partial + m)$ from Left.

$\to (-\not\partial^2 - m^2)\psi = 0$ $\not\partial^2 = \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu = \eta^{\mu\nu}\partial_\mu\partial_\nu = \Box$

So $(\Box + m^2)\psi_\alpha = 0$ for each $\alpha = 1,2,3,4$ each cpt solves KG.

take ansatz $\psi_p(x) = u(p)e^{-ip\cdot x}$ $p^0 = E_p = \sqrt{\vec p^2 + m^2}$. What is $u(p)$

go to rest frame $0 = (i\gamma^\mu\partial_\mu - m)\psi = (i\not\partial - m)u(p)e^{-imt}$

So $(i\gamma^0\partial_t - m)u_p e^{-imt} = 0 \Rightarrow (m\gamma^0 - m)u(p)e^{-mit} = m(\gamma^0 - 1)u(p)e^{-imt}$

$(\gamma^0 - I)\, u_p = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u_p = 0$ so $u(p) = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$. → each is 2 component

21

$\xi_s \quad s = 1, 2. \qquad \xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

$\xi_s^+ \xi_s = 1.$ in general frame

$0 = (i\cancel{\partial} - m)\, u(p) e^{-ip\cdot x} = e^{-ipx}(\cancel{p} - m)\, u(p)$

$(\cancel{p} - m)(\cancel{p} + m) = p^2 - m^2 = 0$ → use this to guess

$u(p) \propto (\cancel{p} + m)$ on some vector

In rest frame $\cancel{p} = m\gamma^0$, $E = m$

$$u_s(p) = \frac{(\cancel{p} + m)}{\sqrt{2(E+m)}} \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}$$

So $u_s(p) \to \dfrac{m\gamma^0 + m}{\sqrt{2m}} \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}$

$= \dfrac{\sqrt{m}}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix} = \sqrt{m}\begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}$

reduces to rest frame soln

Properties

$u_s^+(p)\, u_r(p) = \dfrac{1}{2(E+m)} \cdots$

$$= \frac{1}{2(E+m)} (\xi_s^+ \; \xi_s^+)(\cancel{p} + m)^+ (\cancel{p} + m)\begin{pmatrix} \xi_r \\ \xi_r \end{pmatrix}$$

$$= \frac{1}{2(E+m)} (\xi_s^+ \; \xi_s^+)\begin{pmatrix} m & p\cdot\bar{\sigma} \\ p\cdot\sigma & m \end{pmatrix}\begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix}\begin{pmatrix} \xi_r \\ \xi_r \end{pmatrix}$$

$$= \frac{1}{2(E+m)} (\xi_s^+ \; \xi_s^+)\begin{pmatrix} m^2 + (p\cdot\bar{\sigma})^2 & m(\overbrace{p\cdot\sigma + p\cdot\bar{\sigma}}^{2E}) \\ m(p\cdot\sigma + p\cdot\bar{\sigma}) & m^2 + (p\cdot\sigma)^2 \end{pmatrix}\begin{pmatrix} \xi_r \\ \xi_r \end{pmatrix},$$

Now $(p\cdot\sigma)^2 = p^\mu p^\nu \sigma_\mu \sigma_\nu = (p^0)^2 - 2p^0 \vec{p}\cdot\vec{\sigma} + \underbrace{(\vec{p}\cdot\vec{\sigma})^2}_{\vec{p}^2}$.

$= E^2 - 2E\,\vec{\sigma}\cdot\vec{p} + \vec{p}^2$

Similarly $(p\cdot\bar{\sigma})^2 = E^2 + 2E\,\vec{\sigma}\cdot\vec{p} + \vec{p}^2$

$\bar{\sigma}$ & σ diff in sign of 3 vector parts

So $u_s^+(p)\, u_r(p) = \dfrac{E}{\cancel{2}(E+m)} (\xi_s^+ \; \xi_s^+)\begin{pmatrix} 2E + \cancel{2}\vec{p}\cdot\vec{\sigma} & \cancel{2}m \\ \cancel{2}m & E - \vec{p}\cdot\vec{\sigma} \end{pmatrix}\begin{pmatrix} \xi_r \\ \xi_r \end{pmatrix}$

$= \dfrac{E}{E+m}\, \xi_s^+ \big[\underbrace{E + \cancel{\vec{p}\cdot\vec{\sigma}} + m + m + E - \cancel{\vec{p}\cdot\vec{\sigma}}}_{2(E+m)}\big]\, \xi_r = 2E\, \xi_s^+ \xi_r = 2E\,\delta_{rs}$

$$\boxed{u_s^+(p)\, u_r(p) = 2E_p\, \delta_{rs}}$$

not Lorentz scalar $u_s^+ u_r$ for like other E_p

on other hand this is a Lorentz scalar $\bar{u}_s(p)u_r(p) = \frac{1}{2(E+m)}\begin{pmatrix}\xi_s^+ & \xi_s^+\end{pmatrix}\underbrace{\begin{pmatrix} m & p\cdot\bar\sigma \\ p\cdot\sigma & m\end{pmatrix}\begin{pmatrix} p\cdot\bar\sigma & m \\ m & p\cdot\sigma\end{pmatrix}}\begin{pmatrix}\xi_r \\ \xi_r\end{pmatrix}$

$$\begin{pmatrix} 2m\,p\cdot\sigma & m^2 + p\cdot\bar\sigma\, p\cdot\sigma \\ p\cdot\sigma\, p\cdot\bar\sigma + m^2 & 2m\,p\cdot\sigma \end{pmatrix}$$

What is $(p\cdot\sigma)(p\cdot\bar\sigma) = p^\mu p^\nu \sigma_\mu \bar\sigma_\nu = (p^0)^2 - E\,\cancel{\vec p\cdot\vec\sigma} + E\,\cancel{\vec p\cdot\vec\sigma} - (\vec p\cdot\vec\sigma)^2$

$$= E^2 - \vec p^{\,2} = m^2$$

Now $(p\cdot\bar\sigma)(p\cdot\sigma) = (p\cdot\sigma)(p\cdot\bar\sigma)$ (commute) $= m^2$.

So $\bar u_s(p)u_r(p) = \frac{1}{2(E+m)}\begin{pmatrix}\xi_s^+ & \xi_s^+\end{pmatrix}\begin{pmatrix} 2m\,p\cdot\bar\sigma & 2m^2 \\ 2m^2 & 2m\,p\cdot\sigma\end{pmatrix}\begin{pmatrix}\xi_r \\ \xi_r\end{pmatrix}$.

$$= \frac{1}{2(E+m)}\,\xi_s^+\underbrace{\left(p\cdot\bar\sigma + m + m + p\cdot\sigma\right)}_{2m+2E}\xi_r = 2m\,\delta_{rs}.$$

So $\boxed{\bar U_s(p)\,U_r(p) = 2m\,\delta_{rs}}$

Tomorrow more properties of spinors. & quantize Dirac fld.

Saturday (Refer to Itzykson & Zuber - but they use different γ-repn) $\to$ the $-ip\cdot x$ $\to$ +ve freq.

Recall plane wave sol of Dirac $\psi_p(x) = u(p)\,e^{-ip\cdot x}$ $\quad \xi_1 = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad \xi_2 = \begin{pmatrix}0\\1\end{pmatrix}$

then $(\not p - m)\,u_p = 0$. & $u(p) = \sqrt{m}\begin{pmatrix}\xi_0\\ \xi_0\end{pmatrix}$

for rest frame. More generally, $u_s(p) = \frac{(\not p + m)}{\sqrt{2(E+m)}}\begin{pmatrix}\xi_s\\ \xi_s\end{pmatrix}$

$u_s^+(p)\,u_r(p) = 2E_p\,\delta_{rs}$ and $\bar u_s(p)\,u_r(p) = 2m\,\delta_{rs}$.

for $r = 1,2$ we got 2 linearly indep solns. Next Negative freq. solns

Now try $\psi_p(x) = v(p)e^{ip\cdot x}$ then $(i\not\partial - m)\psi_p(x) = 0$

$\Rightarrow (i\not\partial - m)\,v_p(x)\,e^{ip\cdot x} = 0 \Rightarrow (\not p + m)\,v_p(x) = 0$.

Now solve in rest frame. $(i\gamma^0\partial_0 - m)\,v_p\,\cancel{(x)}\,e^{imt} = 0$

So $(\gamma^0 + 1)\,v_p(x) = 0$ or $\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} v_p\cancel{(x)} = 0$

So in rest frame $v_s(p) = \sqrt{m}\begin{pmatrix}\xi_s\\ -\xi_s\end{pmatrix}$ for same ξ_s as before.

Now go to general frame. as before get $v_s(p) = \frac{-(\not p - m)}{\sqrt{2(E+m)}}\begin{pmatrix}\xi_s\\ -\xi_s\end{pmatrix}$.

check it agrees w/ rest frame $v_s(p) \to \frac{-(m\gamma^0 - m)}{2\sqrt{m}}\begin{pmatrix}\xi_s\\ -\xi_s\end{pmatrix}$

$v_s(p) \to \frac{-\sqrt{m}}{2}\underbrace{(\gamma^0 - 1)}_{\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}}\begin{pmatrix}\xi\\ -\xi\end{pmatrix} = -\frac{\sqrt{m}}{2}\begin{pmatrix}2\xi\\ -2\xi\end{pmatrix} = \sqrt{m}\begin{pmatrix}\xi\\ -\xi\end{pmatrix}$ as desired

Note the odd dimensions of $\psi_p(x)$ $\to M^{1/2}$

These are just soln of a linear Dirac eqn, so can have any dim.
will fix the dim when doing mode expansion

There is another neat way to write check

$$v_s(p) = \frac{-(\not{p}-m)}{\sqrt{2(E+m)}}\begin{pmatrix}\xi_s\\ -\xi_s\end{pmatrix} = \frac{-\gamma^5(\not{p}+m)}{\sqrt{2(E+m)}}\begin{pmatrix}\xi_s\\ \xi_s\end{pmatrix} \overset{?}{=} \frac{-(-\not{p}+m)\gamma^5}{\sqrt{2(E+m)}}\begin{pmatrix}\xi_s\\ \xi_s\end{pmatrix}.$$

as γ^5 anti comm w/ γ^μ, used $-\gamma^5 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$.

So $$v_s(p) = \frac{(-\not{p}+m)}{\sqrt{2(E+m)}}\begin{pmatrix}\xi_s\\ -\xi_s\end{pmatrix} = \frac{-(\not{p}-m)}{\sqrt{2(E+m)}}\begin{pmatrix}\xi_s\\ -\xi_s\end{pmatrix}$$ as desired

useful for calculations & can remember easily from $u_s(p)$.
Calc reduce to those for u_s.

Now check orthogonality

$$v_s^+(p)\,v_r(p) = \frac{1}{2(E+m)}\left(\xi_s^+\ \xi_s^+\right)(\not{p}+m)^+\underbrace{\gamma^5\gamma^5}_{=I}(\not{p}+m)\begin{pmatrix}\xi_r\\ \xi_r\end{pmatrix}.$$

$= 2E_p\,\delta_{rs}$ (calc is same as for $u_s^+ u_r = 2E_p\delta_{rs}$)

Now for the Pauli adjoints.

$$\bar{v}_s(p)\,v_r(p) = \frac{1}{2(E+m)}\left(\xi_s^+\ \xi_s^+\right)(\not{p}+m)^+\underbrace{\gamma^5\gamma^0\gamma^5}_{-\gamma^0}(\not{p}+m)\begin{pmatrix}\xi_r\\ \xi_r\end{pmatrix}$$

$= -2m\,\delta_{rs}$.

So 4 orthogonality relations

$$u_s^+(p)u_r(p) = 2E_p\,\delta_{rs}$$
$$\bar{u}_s(p)\,u_r(p) = 2m\,\delta_{rs}$$
$$v_s^+(p)\,v_r(p) = 2E_p\delta_{rs}$$
$$\bar{v}_s(p)\,v_r(p) = -2m\,\delta_{rs}.$$

<u>We also need</u>

$$\overline{u_r(p)}\,v_s(p) = u_r^+(p)\gamma^0 v_s(p) = \frac{1}{2(E+m)}\left(\xi_r^+\ \xi_r^+\right)(\not{p}+m)^+\gamma^0\gamma^5$$

Now $\gamma^0\gamma^5 = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}-1&0\\0&1\end{pmatrix} = \begin{pmatrix}0&1\\-1&0\end{pmatrix}$; $(\not{p}+m)\begin{pmatrix}\xi_s\\ \xi_s\end{pmatrix}$

$$\Rightarrow \overline{u_r(p)}\,v_s(p) = \frac{1}{2(E+m)}\left(\xi_r^+\ \xi_r^+\right)\begin{pmatrix}m & p\cdot\sigma\\ p\cdot\sigma & m\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}m & p\cdot\sigma\\ p\cdot\sigma & m\end{pmatrix}\begin{pmatrix}\xi_s\\ \xi_s\end{pmatrix}$$

$$= \frac{1}{2(E+m)}\left(\xi_r^+\ \xi_r^+\right)\underbrace{\begin{pmatrix}m & p\cdot\sigma\\ p\cdot\sigma & m\end{pmatrix}\begin{pmatrix}-p\cdot\sigma & -m\\ m & p\cdot\sigma\end{pmatrix}}_{=0}\begin{pmatrix}\xi_s\\ \xi_s\end{pmatrix} = 0$$

So $\boxed{\bar{u}_r(p)\,v_s(p) = 0}$

Next $$u_r^+(\vec{p})\,v_s(-\vec{p}) = \frac{1}{2(E+m)}\left(\xi_r\right.$$

24

$$u_r^\dagger(\vec p)\, v_s(-\vec p) = \frac{-1}{2(E+m)} \begin{pmatrix} \xi_r^\dagger & \xi_r^\dagger \end{pmatrix} (\not p + m)^\dagger \gamma^5 (\underset{\vec p \to -\vec p}{\not p} + m) \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}$$

$$= \frac{1}{2(E+m)} \begin{pmatrix} \xi_r^\dagger & \xi_r^\dagger \end{pmatrix} \begin{pmatrix} m & p\cdot\bar\sigma \\ p\cdot\sigma & m \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} m & p\cdot\bar\sigma \\ p\cdot\sigma & m \end{pmatrix}}_{\begin{pmatrix} m & p\cdot\bar\sigma \\ -p\cdot\sigma & -m \end{pmatrix}} \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}.$$

$$= 0$$

Similarly $v_s^\dagger(p)\, u_r(-\vec p) = 0$. (non zero if you consider $u_r^\dagger(\vec p)\, v_s(\vec p) \neq 0$.

Spin Sums 4x4 matrix.

$$\sum_{S=1,2} u_s(p)\, \bar u_s(p) = \frac{1}{2(E+m)} \sum_s (\not p + m) \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix} \begin{pmatrix} \xi_s^\dagger & \xi_s^\dagger \end{pmatrix} (\not p + m)^\dagger \gamma^0.$$

Since ~~Now~~ $\xi_s \xi_s^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}$. identity

Note: $\sum_s \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix} \begin{pmatrix} \xi_s^\dagger & \xi_s^\dagger \end{pmatrix} = \begin{pmatrix} \xi_s\xi_s^\dagger & \xi_s\xi_s^\dagger \\ \xi_s\xi_s^\dagger & \xi_s\xi_s^\dagger \end{pmatrix} = \begin{pmatrix} I & I \\ I & I \end{pmatrix}.$

$$\text{So } \sum_s u_s(p)\, \bar u_s(p) = \frac{1}{2(E+m)} (\not p + m) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} (\not p + m)^\dagger \gamma^0$$

$$\begin{pmatrix} m & p\cdot\bar\sigma \\ p\cdot\sigma & m \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p\cdot\bar\sigma & m \\ m & p\cdot\sigma \end{pmatrix}$$

So

$$\sum_s u_s(p)\, \bar u_s(p) = \frac{1}{2(E+m)} \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar\sigma & m \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} p\cdot\bar\sigma & m \\ m & p\cdot\sigma \end{pmatrix}$$

$$= \frac{1}{2(E+m)} \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar\sigma & m \end{pmatrix} \begin{pmatrix} m + p\cdot\bar\sigma & m + p\cdot\sigma \\ m + p\cdot\bar\sigma & m + p\cdot\sigma \end{pmatrix}$$

$$= \frac{1}{2(E+m)} \begin{pmatrix} 2m^2 + m\overbrace{(p\cdot\sigma + p\cdot\bar\sigma)}^{2E} & m^2 + 2m\, p\cdot\sigma + (p\cdot\sigma)^2 \\ m^2 + 2m\, p\cdot\bar\sigma + (p\cdot\bar\sigma)^2 & 2m^2 + m\underbrace{(p\cdot\bar\sigma + p\cdot\sigma)}_{2E} \end{pmatrix}$$

$$= \frac{1}{2(E+m)} \begin{pmatrix} 2(m^2 + Em) & m^2 + 2m(E - \vec p\cdot\vec\sigma) + E^2 + \vec p^{\,2} - 2E(\vec p\cdot\vec\sigma) \\ m^2 + 2m(E + \vec p\cdot\vec\sigma) + E^2 + \vec p^{\,2} + 2E\,\vec p\cdot\vec\sigma & 2(m^2 + Em) \end{pmatrix}.$$

$$2E^2 + 2mE - 2m\,\vec p\cdot\vec\sigma - 2E\,\vec p\cdot\vec\sigma$$
$$= 2E(E+m) - 2(E+m)\,\vec p\cdot\vec\sigma$$
$$= 2(E+m)(E - \vec p\cdot\vec\sigma).$$

$$= \frac{1}{\cancel{2(E+m)}} \begin{pmatrix} \cancel{2}m\cancel{(E+m)} & \cancel{2(E+m)}(E - \vec p\cdot\vec\sigma) \\ \cancel{2(E+m)}(E + \vec p\cdot\vec\sigma) & \cancel{2}m\cancel{(E+m)} \end{pmatrix} = \not p + m.$$

So Remarkably: $\boxed{\sum_s u_s(p)\, \bar u_s(p) = \not p + m}$

Similarly

$$\sum_s v_s(p)\overline{v_s}(p) = \frac{1}{2(E+m)} \sum_s \gamma^5 (\not p + m) \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix} (\xi_s^+ \ \xi_s^+)(\not p + m)^+ \underbrace{\gamma^5 \gamma^0}_{-\gamma^0\gamma^5}$$

aside from γ^5 on Left & right, this is same as before, so

$$= -\gamma^5(\not p + m)\gamma^5 = -(\gamma^5)^2(-\not p + m) = \not p - m.$$

$$\boxed{\text{So } \sum_s u_s(p)\overline{u}_s(p) = \not p + m \quad \text{and} \quad \sum_s v_s(p)\overline{v_s}(p) = \not p - m}$$

Mode expansion of quantum Dirac field

$$\psi(x) = \int \frac{[d^3p]}{\sqrt{2E_p}} \sum_s \left[a_s(\vec p)\, u_s(p)\, e^{-ip\cdot x} + b_s^+(\vec p)\, v_s(p)\, e^{ip\cdot x} \right] \Big|_{p^0 = E_p}$$

If we were expanding a Mayorana field then $b_s^+ = a_s^+$.
as it is a real, self charge conjugate field.

Canonical quantization: Fermions anticommute for Pauli pple
$\psi(x)$ anticommuting fields. propose anticomm relns for
a, b, a^+, b^+:

$$\{a_s(p), a_r^+(q)\} = (2\pi)^3 \delta^3(\vec p - \vec q)\, \delta_{rs}.$$

$$\{b_s(p), b_r^+(q)\} = \quad ''$$

a's & b's anticommut. Use these to calculate

$$\{\psi_\alpha(\vec x, t), \psi_\beta^+(\vec y, t)\} = \int \frac{[d^3p]}{\sqrt{2E_p}} \frac{[d^3q]}{\sqrt{2E_q}} \sum_{rs} \{ a_s(\vec p)\, u_{s\alpha}(p)\, e^{-ip\cdot x}$$

$$+ b_s^+(p)\, v_{s\alpha}(p)\, e^{ip\cdot x},\ a_r^+(\vec q)\, u_{r\beta}^+(q)\, e^{+iq\cdot y} + b_r(\vec q)\, v_{r\beta}^+(\vec q)\, e^{-iq\cdot y}\}$$

only 2 non trivial contributions.

$$\left\{ \begin{aligned} &\{a_s(\vec p), a_r^+(\vec p)\}\, u_{s\alpha}(p)\, v_{r\beta}^+(q)\, e^{-ip\cdot x + iq\cdot y} \\ &+ \{b_r(\vec q), b_s^+(\vec p)\}\, v_{s\alpha}(p)\, v_{r\beta}^+(q)\, e^{ip\cdot x - iq\cdot y} \\ &= (2\pi)^3 \delta^3(\vec p - \vec q)\, \delta_{rs} \left[u_{s\alpha}(p)\, u_{r\beta}^+(q)\, e^{-ip\cdot x + iq\cdot y} + v_{s\alpha}(p)\, v_{r\beta}^+(q)\, e^{ip\cdot x - iq\cdot y} \right] \end{aligned} \right.$$

So

$$\{\psi_\alpha(\vec x, t), \psi_\beta^+(\vec y, t)\} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \Big(u_{s\alpha}(\vec p)\, u_{s\beta}^+(\vec p)\, e^{i\vec p\cdot(\vec x - \vec y)} + v_s(\vec p)\, v_{r\beta}^+(\vec p)\, e^{-i\vec p(\vec x - \vec y)} \Big).$$

Now $u^+ = \bar u \gamma^0$ and $v^+ = \bar v \gamma^0$

$$\{\psi_\alpha, \psi_\beta^+\} = \int \frac{[d^3p]}{2E_p} \sum_s \Big(\underbrace{u_{s\alpha}(p)\bar{u}_{s\gamma}(\bar{p})}_{\text{Spin sum}} e^{i\vec{p}\cdot(x-y)} + \underbrace{v_{s\alpha}(p)\bar{v}_{s\gamma}(\bar{p})}_{\text{Spin sum}} e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\Big)(\gamma^0)_{\gamma\beta}$$

26

$$= \int \frac{[d^3p]}{2E_p}\left[(\not{p}+m)_{\alpha\gamma} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + (\not{p}-m)_{\alpha\gamma} e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\right](\gamma^0)_{\gamma\beta}.$$

take $\vec{p}\to-\vec{p}$ in 1st integral.

$$= \int \frac{[d^3p]}{2E_p} e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\left[(\gamma^0p^0 + \vec{\gamma}\cdot\vec{p} \underset{+m}{})_{\alpha\gamma} + (\gamma^0p_0 - \cancel{\vec{\gamma}\cdot\vec{p}} - \cancel{m})_{\alpha\gamma}\right](\gamma^0)_{\gamma\beta}.$$

$$= \delta_{\alpha\beta}\int [d^3p] \frac{e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}}{2E_p} 2E_p = \delta^3(\vec{x}-\vec{y})\,\delta_{\alpha\beta}.$$

So we have shown $\{\psi_\alpha(\vec{x},t), \psi_\beta(\vec{y},t)\} = \delta_{\alpha\beta}\,\delta^3(\vec{x}-\vec{y})$.
can. anti-commutation relations

Next we want $\Pi_\psi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}}$ and $\Pi_{\bar{\psi}} = \frac{\partial\mathcal{L}}{\partial\dot{\bar{\psi}}}$ $\quad \mathcal{L} = \bar{\psi}(i\not{\partial}-m)\psi.$

$$\frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \bar{\psi}i\gamma^0 = i\psi^+. \qquad \frac{\partial\mathcal{L}}{\partial\dot{\bar{\psi}}} = 0.$$

So $\{\psi_\alpha(x,t), \Pi_\beta(\vec{y},t)\} = i\{\psi_\alpha\ \psi_\beta^+\} = i\delta_{\alpha\beta}\,\delta^3(\vec{x}-\vec{y})$

Note: can write $\mathcal{L}_{new} = \frac{1}{2}\left(\bar{\psi}(i\not{\partial}-m)\psi - \bar{\psi}(i\overleftarrow{\not{\partial}}+m)\psi\right)$ but this is related to $\mathcal{L}$ via integration by parts. but there will be more non-zero qtys.

$x_\nu \to x_\nu + \epsilon_\nu$
$\delta_\nu\psi = \partial_\nu\psi.$

Stress Tensor $\quad T_{\mu\nu} = -\eta_{\mu\nu}\mathcal{L} + \underbrace{\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}}_{\bar{\psi}i\gamma^\mu}\partial_\nu\psi.$

$\mathcal{L} = \bar{\psi}(i\not{\partial}-m)\psi.$ $\quad$ ~~$T_{\mu\nu} = -\eta_{\mu\nu}\mathcal{L}$~~

$T_{\mu\nu} = \underset{\text{ignore}}{\cancel{-\eta_{\mu\nu}\mathcal{L}}} + i\bar{\psi}\gamma_\mu\partial_\nu\psi$ $\quad$ This is not symmetric but can make it symm (Belinfante) $\longrightarrow \mathcal{L}=0$ on shell (when eom satisfied)

$$\tilde{T}_{\mu\nu} = \frac{i}{4}\left[\bar{\psi}\gamma_\mu\partial_\nu\psi + \bar{\psi}\gamma_\nu\partial_\mu\psi - (\partial_\nu\bar{\psi})\gamma_\mu\psi - \partial_\mu\bar{\psi}\gamma_\nu\psi\right].$$

taking T_{00} we find hamiltonian $\boxed{H = i\int d^3\vec{x}\ \psi^+\dot{\psi}}$

$T_{\mu\nu} = i\bar{\psi}\gamma_\mu\partial_\nu\psi$ check $\partial_\mu T^{\mu\nu} = i\partial_\mu\bar{\psi}\gamma^\mu\partial_\nu\psi + i\bar{\psi}\gamma^\mu\partial_\nu\partial_\mu\psi.$

$$\partial_\mu T^{\mu\nu} = i\bar{\psi}\overleftarrow{\not{\partial}}\partial_\nu\psi = -m\bar{\psi}\partial_\nu\psi$$
$$+ i\bar{\psi}\partial_\nu\not{\partial}\psi = +m\bar{\psi}\partial_\nu\psi$$
$$= 0.$$

Since $i\not{\partial}\psi = m\psi$
$i\bar{\psi}\overleftarrow{\not{\partial}} = -m\bar{\psi}$

Now write out Hamiltonian in terms of creation-annihilation operators.

$$H = i\int d^3x\, \psi^\dagger \dot\psi = i\int d^3x \int \frac{[d^3p]}{\sqrt{2E_p}} \frac{[d^3q]}{\sqrt{2E_q}} \sum_{rs} \left(a_s^\dagger(p) u_s^\dagger(p) e^{ip\cdot x} \right.$$

$$\left. + b_s(\vec p)\, v_s^\dagger(p)\, e^{-ip\cdot x}\right)(-iE_q)\left(a_r(\vec q) u_r(q) e^{-iq\cdot x} - b_r^\dagger(q) v_r(q) e^{iq\cdot x}\right)$$

$$(2\pi)^3 \; a_s^\dagger(\vec p)\, u_s^\dagger(p)\, \delta^3(\vec p - \vec q)(-iE_q)\, a_r(\vec q)\, u_r(q)$$

$$(2\pi)^3\; b_s(\vec p)\, v_s^\dagger(p)\,(-iE_q)\,\delta^3(p-q)\, b_r^\dagger(\vec q)\, v_r(q).$$

The $u_s^\dagger(\vec p)\, v_r(-p) = 0$

& $v_s^\dagger(p)\, u_r(\underset{-p}{q}) = 0.$

$$= (2\pi)^3\delta^3(\vec p - \vec q)\,(-iE_q)\left(a_s^\dagger(p)\, a_r(q)\, u_s^\dagger(p)\, u_r(q) - b_s(\vec p)\, b_r^\dagger(\vec q)\, v_s(p)\, v_r(q)\right).$$

So $$H = \int \frac{[d^3p]}{2E_p} E_p \sum_{rs}\left(a_s^\dagger(p)\, a_r(p) \underbrace{u_s^\dagger(p) u_r(p)}_{2E_p\delta_{rs}} - b_s(\vec p)\, b_r^\dagger(\vec p) \underbrace{v_s^\dagger(p) v_r(p)}_{2E_p\delta_{rs}}\right)$$

So $$H = \int [d^3p]\, E_p \sum_s \left(a_s^\dagger(\vec p)\, a_s(\vec p) - b_s(\vec p)\, b_s^\dagger(\vec p)\right)$$

Now if you normal order.

$$\boxed{:H: = \int [d^3p]\, E_p \sum_s \left(a_s^\dagger(p) a_s(p) + b_s^\dagger(p) b_s(p)\right)}$$

Now construct basis for Fock space. a & b anti-commute

$$[H, a_s^\dagger(p)] = \int [d^3q]\, E_q \sum_r \left[a_r^\dagger(\vec q)\, a_r(\vec q),\; a_s^\dagger(\vec p)\right]$$

$$\cancel{a_r^\dagger(q) a_r(q) a_s^\dagger(p)} - \underbrace{a_s^\dagger(p)\, a_r^\dagger(q)\, a_r(\vec q) + a_r^\dagger(\vec q)\, a_s^\dagger(\vec p)\, a_r(\vec q)}$$

those terms won't contribute.

$$\underbrace{\qquad}_{(2\pi)^3\delta^3(\vec p - q)\delta_{rs} - \cancel{a_r(\vec q) a_s^\dagger(p)}}$$

So $[H, b_s^\dagger(\vec p)] = E_p\, b_s^\dagger(p)$ | $[H, a^\dagger(p)] = E\, a^\dagger(p)$

$[H, b_s(p)] = -E_p\, b_s(\vec p)$ | $[H, a_s(p)] = -E\, a_s(p)$

<u>Vacuum:</u> $|0\rangle$ $\quad a_s(\vec p)|0\rangle = 0 \quad b_s(\vec p)|0\rangle = 0$

2 kinds of "1-particle" states $a_s^\dagger(\vec p)|0\rangle$ & $b_s^\dagger(p)|0\rangle$

have same egy $\sqrt{|\vec p|^2 + m^2}$. 1 particle 1 antiparticle

3-momentum $\vec{P} = -i\int d^3x\, \psi^\dagger \nabla \psi$.

both the 1 particle & 1 anti particle state have mom $\vec{p}$.

anti comm of a's & b's $\Rightarrow$ Fermi-Dirac statistics.

Normalization $|\vec{p}, s\rangle = \sqrt{2E_p}\, a_s^\dagger(p)\, |0\rangle$

$|\vec{q}, s\rangle = \sqrt{2E_q}\, b_s^\dagger(p)\, |0\rangle$

Tomorrow Sunday 930 am. Spin of Dirac particle spin $\frac{1}{2}$.

Charge of 1 particle states are opposite w/r to $U(1)_V$ charge

after that Maxwell field quantization. then Dirac field propagator.

Sunday 26 Aug. 7th Lecture

Quantization of Dirac

$$\psi(x) = \int \frac{[d^3p]}{\sqrt{2E_p}} \sum_s [a_s(\vec{p}) u_s(p) e^{-ip\cdot x} + b_s^\dagger(p)\, v_s(\vec{p})\, e^{ip\cdot x}]\Big|_{p^0 = E_p}$$

Impose canonical anticomm rel. $\{a_s(\vec{p}), a_r^\dagger(\vec{q})\} = (2\pi)^3 \delta^3(\vec{p}-\vec{q})\delta_{rs}$ etc

This implies $\{\psi_\alpha(\vec{x},t), \Pi_\beta(\vec{y},t)\} = i\delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y})$.

Spin of Dirac field:

Lorentz tr $\delta x^\mu = \omega^\mu{}_\nu x^\nu$ cause change in field.

$\psi'(x') = \exp[-\frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}]\psi(x)$ $S_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$. Then for small ω

$\psi'(x + \omega^{\mu\nu}x_\nu) = (1 - \frac{i}{2}\omega^{\mu\nu}S_{\mu\nu})\psi(x)$

$\psi'(x) + \omega^{\mu\nu}x_\nu\partial_\mu\psi(x)$ $\omega^{\mu\nu}x_\mu\partial_\nu\psi(x) - \frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}\psi(x)$

So $\delta\psi(x) = \psi'(x) - \psi(x) =$

Now $\delta S = \int d^4x\, \partial_\mu [\delta x^\mu \mathcal{L} + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi] \to \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\delta\psi$.

$$0 = \delta S = \int d^4x\, \partial_\mu [\omega^{\mu\nu}x_\nu\mathcal{L} + \bar{\psi} i\gamma^\mu\{\omega^{\alpha\beta}(x_\alpha\partial_\beta) - \frac{i}{2}\omega^{\alpha\beta}S_{\alpha\beta}\}\psi].$$

$$= \int d^4x\, \frac{\omega^{\alpha\beta}}{2}\partial_\mu \underbrace{[(\delta^\mu_\alpha x_\beta - \delta^\mu_\beta x_\alpha)\mathcal{L} + i\bar{\psi}\gamma^\mu(x_\alpha\partial_\beta - x_\beta\partial_\alpha - \frac{i}{2}S_{\alpha\beta})\psi]}_{-M^\mu{}_{\alpha\beta}}$$

true for any $\omega^{\alpha\beta}$ so

$\Rightarrow \boxed{\partial_\mu M^\mu{}_{\alpha\beta} = 0}$. where $\boxed{\eta^\mu{}_\alpha = \delta^\mu_\alpha}$

$$-M^\mu{}_{\alpha\beta} = (\eta^\mu_\alpha x_\beta - \eta^\mu_\beta x_\alpha)\mathcal{L} + i\bar{\psi}\gamma^\mu(x_\alpha\partial_\beta - x_\beta\partial_\alpha)\psi + i\bar{\psi}\gamma^\mu S_{\alpha\beta}\psi$$

$$= x_\beta(\delta^\mu_\alpha\mathcal{L} - i\bar{\psi}\gamma^\mu\partial_\alpha\psi) - x_\alpha(\delta^\mu_\beta\mathcal{L} - i\bar{\psi}\gamma^\mu\partial_\beta\psi) + \bar{\psi}\gamma^\mu S_{\alpha\beta}\psi$$

Recall $T_{\mu\nu} = -\eta_{\mu\nu}\mathcal{L} + i\bar{\psi}\gamma_\mu\partial_\nu\psi$

So $-M^{\mu}{}_{\alpha\beta} = -x_\beta T^{\mu}{}_{\alpha} + x_\alpha T^{\mu}{}_{\beta} + \bar\Psi \gamma^\mu S_{\alpha\beta} \Psi$.

or $-M_{\mu\alpha\beta} = -x_\beta T_{\mu\alpha} + x_\alpha T_{\mu\beta} + \bar\Psi \gamma_\mu S_{\alpha\beta}\Psi$.

To understand spin, look at charge $\int M^0{}_{ij}\, d^3x$

$$J_{ij} = \int d^3x\, M_{0ij} = -\int d^3x\, [-x_j T_{0i} + x_i T_{0j} + \bar\Psi \gamma_0 S_{ij} \Psi]$$

$T_{0i} = \bar\Psi \gamma_0 \partial_i \Psi = \Psi^\dagger \partial_i \Psi$. $\leftarrow$ recall. so

$$J_{ij} = -\int d^3x\, [-x_j \Psi^\dagger \partial_i \Psi + x_i \Psi^\dagger \partial_j \Psi + \Psi^\dagger S_{ij} \Psi]$$

Now define $J_i = \frac{1}{2}\epsilon_{ijk} J_{jk} \equiv +\frac{1}{2}\epsilon_{ijk} \int d^3x [-x_j \Psi^\dagger \partial_k \Psi + x_k \Psi^\dagger \partial_j \Psi$
$+ \Psi^\dagger S_{jk}\Psi]$

$$J_i = -\epsilon_{ijk}\int d^3\vec{x}\; x_j \Psi^\dagger \partial_k \Psi - \frac{1}{2}\epsilon_{ijk}\int d^3x\, \Psi^\dagger S_{jk}\Psi.$$

recall $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \Rightarrow S^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix}\sigma^k & \\ 0 & \sigma^k\end{pmatrix}$

So $J_i = -\epsilon_{ijk}\int d^3x\; x_j \Psi^\dagger \partial_k \Psi - \frac{1}{2}\underbrace{\epsilon_{ijk}\epsilon^{jkl}}_{-2\delta_i^l}\int d^3\vec{x}\, \Psi^\dagger \begin{pmatrix}\sigma^l & 0\\ 0 & \sigma^l\end{pmatrix}\Psi \frac{1}{2}$.

<u>Note</u>
We must have
$\epsilon_{ijk}\epsilon^{jkl} = c\,\delta_i^l$ fixed by $\epsilon_{ijk}\epsilon^{jki} = c = 2\epsilon_{123}\epsilon^{123} = -2$
(3 (-1)'s to bring 3 indices down).

$$J_i = -\epsilon_{ijk}\int d^3x\; x_j \Psi^\dagger \partial_k \Psi + \frac{1}{2}\int d^3x\, \Psi^\dagger \begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}\Psi.$$

So $\boxed{\vec{J} = \int d^3x\; \vec{r} \times (\Psi^\dagger \nabla \Psi) + \int d^3\vec{x}\; \Psi^\dagger \frac{\vec\Sigma}{2} \Psi}$

sign; ~~orbital~~ Mom density; spin density.

$x^\mu = (x^0, x^i)$ | $\partial^i = \frac{\partial}{\partial x_i}$

orbital angular momentum density

To Focus on spin, go to rest frame :: Now to go Schrodinger picture put $\boxed{t=0}$

Now $\Psi(\vec{x}) = \int \frac{[d^3p]}{\sqrt{2E_p}} \sum_s [a_s(\vec{p})\, u_s(p)\, e^{i\vec{p}\cdot\vec{x}} + b_s^\dagger(\vec{p})\, v_s(p)\, e^{-i\vec{p}\cdot\vec{x}}]$

$\left\{\begin{matrix}\text{take } \vec{p} \to -\vec{p} \\ \text{in 2nd term}\end{matrix}\right\}$ $\Psi(\vec{x}, t=0) = \int \frac{[d^3p]}{\sqrt{2E_p}} \sum_s [a_s(\vec{p})\, u_s(\vec{p}) + b_s^\dagger(-\vec{p})\, v_s(-\vec{p})]\, e^{i\vec{p}\cdot\vec{x}}$

So $\vec{J} = \frac{1}{2}\int d^3x \int \frac{[d^3p]}{\sqrt{2E_p}}\, e^{-i\vec{p}\cdot\vec{x}} \sum_r (a_r^\dagger(\vec{p})\, u_s^\dagger(\vec{p}) + b_r(-\vec{p})\, v_r^\dagger(-\vec{p}))\, \vec\Sigma$ | ignore orbital contrib

$\int \frac{[d^3q]}{\sqrt{2E_q}}\, e^{i\vec{q}\cdot\vec{x}} \sum_s (a_s(\vec{q})\, u_s(\vec{q}) + b_s^\dagger(-q)\, v_s(-\vec{q})))$

The d^3x integ gives $\delta^3(\vec{p}-\vec{q})\,(2\pi)^3$

Spin of Dirac field

$$\vec{J} = \frac{1}{2}\int \frac{[d^3p]}{2E_p} \sum_{rs} \left(a_r^\dagger(\vec p)\, v_r^\dagger(\vec p) + b_r(-\vec p)\, v_r^\dagger(-\vec p)\right) \Sigma \left(a_s(\vec p)\, u_s(\vec p) + b_s^\dagger(-\vec p)\, v_s(-\vec p)\right).$$

Normal order this

Look at J^3 should be $\pm\frac{1}{2}$. on a 1 particle state.

$$J^3 a_s^\dagger(\vec 0)|0\rangle = [J^3, a_s^\dagger(\vec 0)]|0\rangle$$

if you normal order. these are same

$$[J^3, a_s^\dagger(\vec 0)]|0\rangle = \frac{1}{2}\int\frac{[d^3p]}{2E_p}\sum_{r's'}\Big[\left(a_{r'}^\dagger(\vec p)\, u_{r'}^\dagger(\vec p) + b_{r'}(-\vec p)\, v_{r'}^\dagger(-\vec p)\right)\Sigma^3 \left(a_{s'}(\vec p)\, u_{s'}(\vec p) + b_{s'}^\dagger(-\vec p)\, v_{s'}(-\vec p)\right), a_s^\dagger(0)\Big]|0\rangle$$

zero mom.

Now look @ terms

$$[a_{r'}^\dagger(\vec p)\, b_{s'}^\dagger(-\vec p), a_s^\dagger(0)] = a_{r'}^\dagger b_{s'}^\dagger a_s^\dagger - a_s^\dagger b_{s'}^\dagger a_{r'}^\dagger \quad a_{r'}^\dagger b_{s'}^\dagger a_s^\dagger = 0.$$

$$[b_{r'}(-\vec p)\, b_{s'}^\dagger(-\vec p), a_s^\dagger(0)] = b_{r'} b_{s'}^\dagger a_s^\dagger - a_s^\dagger b_{r'} b_{s'}^\dagger \quad a_s^\dagger = 0.$$

$$[b_{r'}(-\vec p)\, a_{s'}(\vec p), a_s^\dagger(\vec 0)] = b_{r'}(-\vec p)\, a_{s'}(\vec p)\, a_s^\dagger(0) - \underbrace{a_s^\dagger(0)\, b_{r'}(-\vec p)\, a_{s'}(\vec p)}$$

$$+\, b_{r'}(-\vec p)\, \underbrace{a_s^\dagger(\vec 0)\, a_{s'}(\vec p)}_{(2\pi)^3\delta^3(\vec p)\delta_{ss'} - a_{s'}(\vec p)\, a_s^\dagger(0)}$$

$$\Rightarrow (2\pi)^3\, \delta^3(\vec p)\, \delta_{ss'}\, b_{r'}(-\vec p)$$

↳ will annihilate vac.

So get $v_{r'}^\dagger(\vec 0)\, \Sigma^3\, v_s(\vec 0)$ use rest frame expressions for v spinors.

//

$$\left(\xi_{r'}^\dagger \;\; -\xi_{r'}^\dagger\right)\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix}\begin{pmatrix}\xi_s\\ -\xi_s\end{pmatrix}$$

One more term to consider: $[a_{r'}^\dagger(\vec p)\, a_{s'}(\vec p), a_s^\dagger(\vec 0)]\,|0\rangle$

$$= \{a_{r'}^\dagger(\vec p)\, a_{s'}(\vec p)\, a_s^\dagger(\vec 0) - a_s^\dagger(0)\, a_{r'}^\dagger(\vec p)\, \cancel{a_{s'}(\vec p)}\}\,|0\rangle.$$

↓replace w/ $\{a_{s'}(\vec p), a_s^\dagger(0)\}$ as acting on $|0\rangle$

$$= (2\pi)^3\, \delta^3(\vec p)\, \delta_{ss'}\, a_{r'}^\dagger(\vec p)\,|0\rangle$$

<u>So get</u>: $[J^3, a_s^\dagger(0)]|0\rangle = \frac{1}{2}\cdot\frac{1}{2m}\sum_{r'} a_{r'}^\dagger(0)|0\rangle\; u_{r'}^\dagger(0)\, \frac{\Sigma^3}{2}\, u_s(0)$

$$u_{r'}^\dagger(\vec 0)\,\frac{\Sigma^3}{2}\,u_s(\vec 0) = m\left(\xi_{r'}^\dagger \;\; \xi_{r'}^\dagger\right)\frac{1}{2}\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix}\begin{pmatrix}\xi_s\\ \xi_s\end{pmatrix} = m\, \xi_{r'}^\dagger\, \sigma^3\, \xi_s$$

↳ from normalization of u spinors in rest frame.

So $J^3 a_s^\dagger(\vec 0)|0\rangle = \frac{1}{2}\sum_r\left(\xi_r^\dagger\, \sigma^3\, \xi_s\right) a_r^\dagger(\vec 0)\,|0\rangle$

look at $s=1$ $\qquad J^3 a_1^+(\vec{0})|0\rangle = \frac{1}{2}\sum_r \xi_r^+ \sigma^3 \xi_1)\ a_r^+(\vec{0})|0\rangle$ (31)

$= \frac{1}{2}(1\ 0)\sigma^3\begin{pmatrix}1\\0\end{pmatrix} a_1^+(0)|0\rangle + \cancel{\frac{1}{2}(0\ 1)\sigma^3\begin{pmatrix}1\\0\end{pmatrix} a_2^+(\vec{0})|0\rangle}$

0 as σ^3 is diagonal

So $\boxed{J^3 a_1^+(\vec{0})|0\rangle = \frac{1}{2} a_1^+(\vec{0})|0\rangle}$ So single particle state w/ $s=1$ has J^3 eval $+\frac{1}{2}$.

Similarly $J^3 a_2^+(\vec{0})|0\rangle = \frac{1}{2}\sum_r \xi_r^+\sigma^3\xi_2\ a_r^+(\vec{0})|0\rangle$.

$= \cancel{\frac{1}{2}(1\ 0)\sigma^3\begin{pmatrix}0\\1\end{pmatrix} a_1^+(\vec{0})|0\rangle} + \frac{1}{2}(0\ 1)\sigma_3\begin{pmatrix}0\\1\end{pmatrix} a_2^+(\vec{0})|0\rangle$

0

So $\boxed{J^3 a_2^+(\vec{0})|0\rangle = -\frac{1}{2} a_2^+(\vec{0})|0\rangle.}$ So ~~[illegible]~~ $s=2$ state has eval of J^3 $-\frac{1}{2}$ Spin down.

Similarly look @ $b_1^+(\vec{0})|0\rangle$ and $b_2^+(\vec{0})|0\rangle$ 1-antiparticle state

$J_3\, b_s^+(\vec{0})|0\rangle = \int\frac{[d^3p]}{2E_p}\sum_{r's'} \underbrace{[b_{r'}(-\vec{p})\, b_{s'}^+(-\vec{p}), b_s^+(\vec{0})]}_{(2\pi)^3\delta^3(\vec{0})\,\delta_{r's'} - b_{s'}^+(-\vec{p})\, b_{r'}(-\vec{p})}|0\rangle\ v_{r'}^+(-\vec{p})\frac{\Sigma^3}{2} v_{s'}(-\vec{p})$

$v_{s'}(-\vec{p})$

This sign is the difference from previous case

So we get

$J^3 b_s^+(\vec{0})|0\rangle = -\sum_{s'}\frac{b_{s'}^+(\vec{0})}{2m}|0\rangle\ v_s^+(\vec{0})\frac{\Sigma^3}{2} v_{s'}(\vec{0})$

$= -\frac{1}{2}\sum_r \xi_s^+\sigma^3\xi_r\ b_r^+(\vec{0})|0\rangle$ $\qquad$ Now take $s=1$ & $s=2$ separately

$$\boxed{\begin{aligned} J^3 b_1^+(\vec{0})|0\rangle &= -\frac{1}{2} b_1^+(\vec{0})|0\rangle \\ J_3 b_2^+(\vec{0})|0\rangle &= \frac{1}{2} b_2^+(\vec{0})|0\rangle \end{aligned}}$$

and So positron also has spin projections ± 1 but $s=1$ has spin^3 $-\frac{1}{2}$ $s=2$ has $\text{spin}^3 = +\frac{1}{2}$.

Charge: $\quad \psi(x) \to e^{i\alpha}\psi(x) \qquad J^\mu = \bar{\psi}\gamma^\mu\psi$.

$Q = \int d^3x\ \psi^+\psi = \int d^3x \int\frac{[d^3\vec{p}]}{\sqrt{2E_p}}\sum_r\Big(a_r^+(\vec{p})\,u_r^+(\vec{p})e^{ip\cdot x} + b_r(\vec{p})v_r^+(\vec{p})e^{-ip\cdot x}\Big)$

$\times \int\frac{[d^3q]}{\sqrt{2E_q}}\sum_s\Big(a_s(\vec{q})u_s(\vec{q})e^{-iq\cdot x} + b_s^+(\vec{q})\ v_s(\vec{q})\ e^{iq\cdot x}\Big)$

4 terms $\quad a^+ b^+$ term$(\vec{q}=-\vec{p})$ $u_r^+(p)\, v_s(-\vec{p}) = 0$ by orthogonality

Similarly $b_r a_s$ term $=0$ by $\perp$.

left w/ 2 terms. with $\delta^3(\vec{p}-\vec{q})$

$$Q = \int \frac{[d^3p]}{2E_p} \sum_{rs} \left[a_r^+(\vec{p})\, u_r^+(\vec{p})\, a_s(\vec{p})\, u_s(\vec{p}) + b_r(p)\, v_r^+(p)\, b_s^+(\vec{p})\, v_s(p) \right]$$

use normalization $u_r^+(\vec{p})\, u_s(\vec{p}) = 2E_p\, \delta_{rs}$. $2E_p$ cancels $\frac{1}{2E_p}$

So $Q = \int [d^3p] \sum_s \left(a_s^+(\vec{p})\, a_s(\vec{p}) + b_s(p)\, b_s^+(\vec{p}) \right)$

Then you normal order

$:Q: = \int d^3\vec{p} \sum_s \left(a_s^+(\vec{p})\, a_s(\vec{p}) - b_s^+(\vec{p})\, b_s(\vec{p}) \right).$

= electron # − positron number.

Now act on 1-particle state

$:Q:\, a_s^+(\vec{p})\,|0\rangle = a_s^+(\vec{p})\,|0\rangle$. and $:Q:\, b_s^+(\vec{p})\,|0\rangle = -b_s^+(\vec{p})\,|0\rangle$.

So properties of single particle states

$|p,s\rangle = \sqrt{2E_p}\; a_s^+(\vec{p})\,|0\rangle$ (particle)

$E = \sqrt{\vec{p}^2+m^2}$, $\hat{\vec{P}} = \vec{p}$, $Q=1$

$J^3 = \begin{cases} s=1 & \frac{1}{2} = J^3 \\ s=2 & -\frac{1}{2} = J^3 \end{cases}$

(antiparticle) $|\vec{q}, \vec{s}\rangle = \sqrt{2E_q}\; b_s^+(\vec{q})\,|0\rangle$. $J^3 = \begin{cases} s=1 & -\frac{1}{2} \\ s=2 & \frac{1}{2} \end{cases}$.

$E = \sqrt{\vec{q}^2+m^2}$, $\vec{P} = \vec{q}$, $Q = -1$. and

Dirac Field propagator $\qquad S = \int d^4x\, \bar{\psi}(i\not{\partial} - m)\,\psi$.

$(i\not{\partial}-m)_{\alpha\beta}\, S_F(x-y) = i\,\delta^4(x-y)\,\mathbb{1}$

$S_F(x-y) = \int [d^4p]\, S_F(\vec{p})\, e^{-ip\cdot(x-y)}$. So.

$\int \frac{d^4p}{(2\pi)^4} (\not{p}-m)\, S_F(\vec{p})\, e^{-ip\cdot(x-y)} = i \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}$

So $(\not{p}-m)\, S_F(p) = i$ $\quad$ so $S_F(p) = \frac{i(\not{p}+m)}{p^2-m^2} = \underbrace{\frac{i}{\not{p}-m}}_{\text{inverse of this matrix}}$.

$S_F(x-y) = \int [d^4p] \frac{i(\not{p}+m)\, e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon} \Big|_{\epsilon\to 0^+}$ $\quad$ same $i\epsilon$ prescription.

$= (i\not{\partial}_x+m) \int [d^4p] \frac{i\, e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon} = (i\not{\partial}_x+m)\, D_F(x-y)$

where D_F was the KG field Causal Feynman propagator.

Now want to write in terms of time ordered products $\langle 0|T\,\phi(x)\,\phi(y)|0\rangle$

(33) $S_F(x-y) = (i\not{\partial}_x + m)\Big[\underbrace{\theta(x^0-y^0)\int \frac{[d^3p]}{2E_p} e^{-ip\cdot(x-y)}}\, + \theta(y^0-x^0)\int\frac{[d^3p]}{2E_p} e^{ip\cdot(x-y)}\Big]$

look @ this 2 contrib. ∂_x hits $\theta(x^0-y^0)$ or the $\int$ ...

$= \theta(x^0-y^0)\,(i\not{\partial}+m)\int\frac{[d^3p]}{2E_p}e^{-ip\cdot(x-y)} + \theta(y^0-x^0)(i\not{\partial}+m)\int\frac{[d^3p]}{2E_p}e^{ip\cdot(x-y)}$ (**)

$+ i\gamma^0 \underbrace{\left[\frac{\partial}{\partial x^0}\theta(x^0-y^0)\right]}_{\delta(x^0-y^0)} \int\frac{[d^3p]}{2E_p}e^{-ip\cdot(x-y)} + i\gamma^0 \underbrace{\frac{\partial\theta(y^0-x^0)}{\partial x^0}}_{-\delta(x^0-y^0)}\int\frac{[d^3p]}{2E_p}e^{ip\cdot(x-y)}$

$\delta(x^0-y^0) \longrightarrow e^{i\vec{p}\cdot(\vec{x}-\vec{y})}$ $\qquad e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}$

time dep cancels

$= $ 1st 2 terms $+ i\gamma^0\int[d^3p]\frac{e^{i\vec{p}\cdot(\vec{x}-\vec{y})}}{2E_p} - i\gamma^0\int[d^3p]\frac{e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}}{2E_p}$

Now take $\vec{p}\to-\vec{p}$ in ~~1st~~ integration variable

last 2 terms cancel.

So only 1st 2 terms survive & have $\theta(x^0-y^0)$ & $\theta(y^0-x^0)$ just like for KG Field. Want to relate to $\langle 0|T(\cdots)|0\rangle$

Now lets calc $\langle 0|\psi_\alpha(x)\bar\psi_\beta(y)|0\rangle$ = particle propagation.

$\langle 0|\psi_\alpha(x)\bar\psi_\beta(y)|0\rangle = \langle 0|\int\frac{[d^3q]}{\sqrt{2E_q}}\sum_r a_r(\vec{q})\,u_{r\alpha}(q)\,e^{-iq\cdot x}\int\frac{d^3p}{\sqrt{2E_p}}\sum_s a_s^\dagger(p)\bar{u}_{s\beta}(p)\, e^{ip\cdot y}|0\rangle$

use anticomm $\quad 2\pi^3\,\delta(\vec{p}-\vec{q})\,\delta_{rs}$.

$= \int\frac{[d^3p]}{2E_p}\langle 0|\underbrace{\sum_s u_{s\alpha}(p)\,\bar{u}_{s\beta}(p)}_{\text{spin sum} = (\not{p}+m)_{\alpha\beta}}|0\rangle\, e^{-ip\cdot(x-y)}$

$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}(\not{p}+m)_{\alpha\beta}\,e^{-ip\cdot(x-y)} = (i\not{\partial}_x+m)_{\alpha\beta}\int\frac{[d^3p]}{2E_p}e^{-ip\cdot(x-y)}$

This matches 1st term in (**) above

Similarly consider antiparticle propagation

$\langle 0|\bar\psi_\beta(y)\psi_\alpha(x)|0\rangle = \langle 0|\int\frac{[d^3q]}{\sqrt{2E_q}}\sum_r b_r(q)\,\bar{v}_{r\beta}(\vec{q})\,e^{iq\cdot y}\int\frac{[d^3p]}{\sqrt{2E_p}}\sum_s b_s^\dagger(\vec{p})\,v_{s\alpha}(\vec{p})\,e^{-ip\cdot x}|0\rangle$

anticomm gives $(2\pi)^3\delta^3(\vec{p}-\vec{q})\,\delta_{rs}$

So $\langle 0|\bar{\Psi}_\beta(y)\,\Psi_\alpha(x)|0\rangle = \int \frac{[d^3p]}{2E_p}\, e^{ip\cdot(x-y)} \underbrace{\sum_s \langle 0|\bar{v}_{s\beta}(p)\, v_{s\alpha}(p)|0\rangle}_{(\not{p}-m)_{\alpha\beta}}$ ↳ use spin sum

$$= \int \frac{[d^3p]}{2E_p} (\not{p}-m)_{\alpha\beta}\, e^{ip\cdot(x-y)} = -(i\not{\partial}+m)_{\alpha\beta} \int [d^3p]\, \frac{e^{ip\cdot(x-y)}}{2E_p}$$

$$\text{So } (S_F(x-y))_{\alpha\beta} = \theta(x^0-y^0)\langle 0|\Psi_\alpha(x)\,\bar{\Psi}_\beta(y)|0\rangle - \theta(y^0-x^0)\langle 0|\bar{\Psi}_\beta(y)\,\Psi_\alpha(x)|0\rangle.$$

$$\equiv \langle 0|T\,\Psi_\alpha(x)\,\bar{\Psi}_\beta(y)|0\rangle$$

where $T(\Psi_1\Psi_2) = \begin{cases} \Psi_1\Psi_2 & x_1^0 > x_2^0 \\ -\Psi_2\Psi_1 & x_2^0 > x_1^0 \end{cases}$.

Next Maxwell Theory quantization. gauge invariance
↳ do it in radiation gauge $A^0=0$ $\nabla\cdot A=0$, not by Gupta-Bleuler method.
Then S-matrix & Wick's theorem.
Feynman diagrams for ϕ^4 & QED (heuristically).
Do not plan to do matrix element & cross-section calculations
No $e^+e^-\to\mu^+\mu^-$ cross-section.
Then go on to Non abelian gauge theory: pure YM and coupling to fermions in various representations. Also plans to do SSB for a U(1) theory Abelian Higgs model

<u>Quantization of the EM field</u>: Monday 27 August

Source-free Maxwell eqns.

$\nabla\cdot B=0 \qquad \nabla\times B = +\frac{\partial E}{\partial t}$

$\nabla\cdot E=0 \qquad \nabla\times E = -\frac{\partial \vec{B}}{\partial t}$

Plane wave ansatz.

$\bar{E}(x,t) = \vec{E}_0\, e^{-i(k\cdot x-\omega t)}$

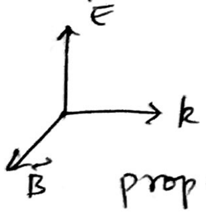
$\bar{B}(x,t) = B_0\, e^{-i(k\cdot x-\omega t)}$

Put in eqns give

$\vec{k}\cdot\vec{B}_0 = 0$

$\vec{k}\cdot\vec{E}_0 = 0$

$\bar{k}\times\vec{B}_0 = -\omega E_0$ and $k\times\vec{E}_0 = \omega\vec{B}_0$ will drop the 0-subscripts.



<u>Suppose $\omega>0$.</u> E & B are $\perp$ to $\vec{k}$ → new feature of Maxwell, not seen in scalar & Dirac field
Degrees of freedom of EM radiation are $\perp$ to direc of propagation. Not all cpts of A^μ are indep Need to impose this constraint. We will use a non covariant wave of quantization & use std canonical quantization. Phys d.o.freedom will be explicit.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\left(2F_{0i}F^{0i} + F_{ij}F^{ij}\right) = -\frac{1}{2}\left(F_{0i}F^{0i}\right) - \frac{1}{4}F_{ij}F^{ij}$$

Canonical mom: $\pi^\mu = \frac{\partial\mathcal{L}}{\partial\dot{A}_\mu} = ?$ $\quad\pi^0 = \frac{\partial\mathcal{L}}{\partial\dot{A}_0} = 0$. So A^0 has no conj mom.

$$\pi^i = \frac{\partial\mathcal{L}}{\partial\dot{A}^i} = -F^{0i} = E^i$$

Thus $\pi^0 = 0$ and $\vec{\pi} = \vec{E}$

Suggests that A^0 may not be a true propagating degree of freedom.

Guess Equal time canonical commutation relations.

expect $[A^\mu(\vec{x},t), A^\nu(\vec{y},t)] = 0$.

$[\pi^i(\vec{x},t), \pi^j(y,t)] = 0$

guess

$[\pi^i(x,t), A^0(\vec{y},t)] = 0$

$[\pi^i(x,t), A^j(\vec{y},t)] \overset{?}{=} + i\delta^{ij}\delta^3(\vec{x}-\vec{y})$ — (4)

plus as expect $[A^\mu, \pi^\nu] \propto \eta^{\mu\nu}$, $\eta^{ij} = -\delta^{ij}$.

Let us check these: by calc

E & B : we get

$[E^i(x,t), A^j(y,t)] = i\delta^{ij}\delta^3(x-y)$

$\Rightarrow [\underbrace{\nabla\cdot E(\vec{x},t)}_{=0}, A^j(y,t)] = i\partial_{x_j}\delta^3(x-y)$

↳ This should be zero

So we need to correct (4). Try:

$$[\pi^i(\vec{x},t), A^j(\vec{y},t)] = i\left(\delta^{ij} - \frac{\partial_x^i\partial_x^j}{\nabla_x^2}\right)\delta^3(x-y) \quad \text{— (4')}$$

Now apply ∂_i^x to this (4'): LHS = 0 as $\nabla\cdot E = 0$.

$$\text{RHS} = i\left(\partial^j_x - \frac{\nabla^2\partial^j_x}{\nabla^2}\right)\delta^3(x-y) = 0$$

on the other hand, suppose we act w/ ∂^j_y:

use $\partial^j_x = -\partial^j_y$ when acting on fns of $x-y$. So

$$\partial^j_y(\text{RHS}) \text{ becomes} = \partial^j_y\left(i\left(\delta^{ij} - \frac{\partial^i_y\partial^j_y}{\nabla_y^2}\right)\delta^3(x-y)\right) = i\left(\partial^i_y - \frac{\partial^i_y\vec{\nabla}_y^2}{\nabla_y^2}\right)\delta^3(x-y)$$
$$= 0$$

So we find that $A^0(\vec{x},t)$ commutes w/ everything.
and $\nabla\cdot A(x,t)$ commutes w/ everything.
} c-numbers need not be "quantized".

$A^0 \to$ temporal cpt of photon field

$\nabla\cdot A$ in k-space is $k\cdot\tilde{A}_k \to$ longitudinal photon mode.

So A^0 & $\nabla\cdot A$ are not propagating deg. of freedom

So we need not worry about quantizing them. Need to focus only on the transverse cpts of the gauge field.

We will make gauge tr to eliminate A^0 & $\vec{\nabla}\cdot\vec{A}$

$A_\mu \to A_\mu + \partial_\mu\Lambda$, $F_{\mu\nu} \to F_{\mu\nu}$.

Gauge transformations: start w/ $A_\mu = (\Phi, \vec{A})$. We will make 2 gauge tr. in succession.
↳ general gauge potn.

① $A_\mu \to A'_\mu$ $\quad A'_\mu(x,t) = A_\mu(x,t) - \partial_\mu \int_0^t dt' \, \Phi(x,t')$

$$\Rightarrow A'_0(x,t) = \Phi(x,t) - \underbrace{\partial_t \int_0^t dt' \, \Phi(x,t')}_{\Phi(x,t)} = 0 .$$

So after this gauge tr, $\boxed{A'_0 = 0}$

② Make another gauge tr. $A'_\mu \to A''_\mu = \vec{A}'(\bar{x},t) + \vec{\nabla}\Lambda(x,t)$.

Λ is chosen so that $\vec{\nabla}\cdot\vec{A}'' = 0$. What is Λ ??

$\nabla\cdot A' + \nabla^2 \Lambda = 0$. So need to solve Poisson's eqn

$$\nabla^2\Lambda(x,t) = -\vec{\nabla}\cdot\vec{A}(x,t) \Rightarrow \Lambda(x,t) = \int \frac{d^3\bar{x}'}{4\pi} \frac{\vec{\nabla}'\cdot A'(x',t)}{|x-x'|}$$

Can check this: $\nabla^2\Lambda(x,t) = \int \frac{d^3x'}{4\pi} \vec{\nabla}'\cdot A'(x',t) \, \underbrace{\nabla^2\left(\frac{1}{|x-x'|}\right)}_{-4\pi\delta^3(x-x')}$.

$$= -\vec{\nabla}\cdot A'(x,t) \checkmark$$
as desired.

by ② we have ensured $\nabla\cdot A'' = 0$

But what about A''_0 ?? $A''_0 = \cancel{A'_0(x,t)}_{0} + \frac{\partial}{\partial t}\Lambda(x,t)$

So if we want $A''_0 = 0$ then need to take $\Lambda(x,t)$ indep of time

is this consistent?. Check:

$$\frac{\partial}{\partial t}\Lambda(x,t) = \int \frac{d^3\bar{x}'}{4\pi} \frac{1}{|x-x'|} \frac{\partial}{\partial t}\vec{\nabla}' A'(x',t)$$
→ This is zero by Gauss' Law.

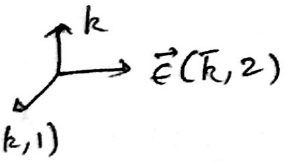
$$\nabla\cdot E' = 0 \Rightarrow \partial_i(\cancel{\partial^i A^0}_0 - \partial^0 A'^i) = 0 \Rightarrow \frac{\partial}{\partial t}(\nabla\cdot A') = 0$$

Thus we can consistently impose both $\boxed{A^0 = 0}$ and $\boxed{\nabla\cdot A = 0}$ at all times. "Radiation Gauge"

This is not Lorentz covariant in a manifest way. but only physical degrees of freedom are present.

We will do mode expansion in this gauge

$\vec{A}$ expand in terms of transverse polarization vectors



by choice, $\bar{k}\cdot\epsilon(\bar{k},\lambda) = 0$ for $\lambda = 1,2$.

$\epsilon(k,\lambda)\cdot\epsilon(k,\lambda') = \delta_{\lambda\lambda'}$ orthonormal system.

Look at $0 = \partial_\mu F^{\mu\nu}$: $\boxed{\nu = 0}$ $\quad 0 = \partial_i F^{i0} = \partial_i(\cancel{\partial^i A^0}_0 - \partial^0 A^i) = -\partial^0 \cancel{\partial_i A^i}_0 = 0$
trivial

take $\boxed{\nu = i}$ $\quad 0 = \partial_\mu F^{\mu i} = \frac{\partial}{\partial t}(F^{0i}) + \partial_j F^{ji} = \partial_t(\partial^0 A^i - \partial^i \cancel{A^0}_0) + \partial_j(\partial^j A^i - \cancel{\partial^i A^j})$

$$= (\partial_t^2 - \nabla^2)A^i = 0$$

So $\boxed{\partial_\mu \partial^\mu \vec{A} = 0}$ in this gauge. every cpt of $\vec{A}$ satisfies mass-less KG eqn. So we can use KG mode exp. 37

$$\vec{A}(\vec{x},t) = \int \frac{[d^3 p]}{\sqrt{2E_p}} \sum_{\lambda=1}^{2} \left[\vec{\epsilon}(\vec{p},\lambda)\, a_\lambda(p) e^{-ip\cdot x} + \vec{\epsilon}(p,\lambda)\, a_\lambda^\dagger(p)\, e^{ip\cdot x} \right] \qquad p^0 = E_p = |\vec{p}|.$$

we are assuming $\vec{\epsilon}$ is real. (could also use cx. polarization vectors). Now $\nabla \cdot A = 0$ is automatically sats as $\vec{p}\cdot\vec{\epsilon}(p,\lambda) = 0$

↳ later for rt & left-circ pol.

Now impose canonical equal time comm relns.

$$[a_\lambda(p), a_{\lambda'}(q)] = 0 = [a_\lambda^\dagger(\vec{p}), a_{\lambda'}^\dagger(\vec{q})].$$

$$[a_\lambda(\vec{p}), a_{\lambda'}^\dagger(\vec{q})] = (2\pi)^3 \delta^3(\vec{p}-\vec{q})\, \delta_{\lambda\lambda'}.$$

$\boxed{\text{Parity}}$ will choose

under parity will suppose polarization vectors transform as

$\epsilon(-k,1) = -\epsilon(k,1)$ and $\epsilon(-k,2) = \epsilon(k,2)$

This will simplify the calculations. This will ensure new $(k, \epsilon(-k,\lambda))$ form an o.n system if $(k, \epsilon(k,\lambda))$ was an o.n. system.

<u>Next calculate</u> $[A^i(x,t), A^j(y,t)] = \int \frac{[d^3 p]}{\sqrt{2E_p}} \int \frac{[d^3 q]}{\sqrt{2E_q}} \sum_{\lambda\lambda'} \epsilon^i(p,\lambda)\epsilon^j(q,\lambda')$

$$\cdot \left[a_\lambda(p) e^{-ip\cdot x} + a_\lambda^\dagger(p) e^{ip\cdot x},\; a_{\lambda'}(q) e^{-iq\cdot y} + a_{\lambda'}^\dagger(q) e^{iq\cdot y} \right]$$

$$= (2\pi)^3 \delta^3(p-q)\delta_{\lambda\lambda'} \;\Big|\; -(2\pi)^3 \delta^3(p-q)\,\delta_{\lambda\lambda'} \qquad \text{plug in \& get}$$

$$[A^i(x), A^j(y)] = \int \frac{[d^3 p]}{2E_p} \sum_\lambda \epsilon^i(\vec{p},\lambda)\epsilon^j(p,\lambda) \left(e^{ip\cdot(x-y)} - e^{-ip\cdot(x-y)} \right) = 0$$

take $\vec{p} \to -\vec{p}$ integ var, use our convention for ϵ under parity

the $\epsilon\epsilon$ factors wont change & the 2 terms cancel.

<u>Next</u> $[\pi^i, \pi^j]$: $\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}^i} = -F^{0i} = -(\partial^0 A^i - \partial^i \cancel{A^0}) = -\frac{\partial A^i}{\partial t}$.

$$\vec{\pi} = -\frac{\partial \vec{A}}{\partial t} = \int \frac{[d^3 p]}{\sqrt{2E_p}} (iE_p) \sum_\lambda \left(a_\lambda(p)\epsilon^i(p,\lambda)\, e^{-ip\cdot x} - a_\lambda^\dagger(p)\, \epsilon^i(\vec{p},\lambda)\, e^{ip\cdot x} \right)$$

So: $[\pi^i(x), \pi^j(y)] = \int \frac{[d^3p\, d^3q]}{\sqrt{2E_p 2E_q}}\, i E_p\, i E_q \sum_{\lambda\lambda'} \epsilon^i(p,\lambda)\, \epsilon^j(\vec q,\lambda')$

$[a_\lambda(\vec p) e^{-ip\cdot x} - a_\lambda^+ e^{ip\cdot x},\ a_{\lambda'}(q) e^{-iq\cdot y} - a_{\lambda'}^+(\vec q)\, e^{iq\cdot y}]$.

$\rightarrow\ -(2\pi)^3 \delta^3(\vec p - q)\, \delta_{\lambda\lambda'} \quad + (2\pi)^3 \delta^3(p-q)\, \delta_{\lambda\lambda'}$

as before get $[\pi^i, \pi^j] = 0$.

Finally calculate:

$[\pi^i(x), A^j(y)] = \int \frac{[d^3p]}{\sqrt{2E_p}} \int \frac{[d^3q]}{\sqrt{2E_q}}\, i E_p\, i E_q \sum_{\lambda\lambda'} \epsilon^i(\vec p,\lambda)\, \epsilon^j(q,\lambda')$

$[a_\lambda(p) e^{-ip\cdot x} - a_\lambda^+(p) e^{ip\cdot x},\ a_{\lambda'}(q) e^{-iq\cdot y} + a_{\lambda'}^+(q)\, e^{iq\cdot y}]$.

$\hookrightarrow (2\pi)^3 \delta^3(p-q)\, \delta_{\lambda\lambda'} \left(e^{i\vec p\cdot(\vec x - \vec y)} \quad + \quad e^{-i\vec p\cdot(\vec x-\vec y)} \right)$

$[\pi^i, A^j] = \frac{i}{2}\int [d^3p] \sum_\lambda \epsilon^i(p,\lambda)\, \epsilon^j(p,\lambda) \left(e^{i\vec p\cdot(\vec x-\vec y)} + e^{-i\vec p\cdot(\vec x-\vec y)} \right)$.

Calc polarization sum $\sum_\lambda \epsilon^i(p,\lambda)\, \epsilon^j(p,\lambda)$ is invariant under $p \to -p$.

2nd rank tensor, symmetric.

So it must be $\delta^{ij} A(\vec p^2) + p^i p^j B(\vec p^2)$ ← most general.

Calc A & B by dotting w/ $\vec p$.

$\lambda = 1$ both pick up sign

$\lambda = 2$ neither get sign

$0 = p^j A(\vec p^2) + \vec p^2 p^j B(\vec p^2) \Rightarrow p^2 B(p^2) + A(p^2) = 0$

So $\sum_\lambda \epsilon^i(\vec p,\lambda)\, \epsilon^j(p,\lambda) = A(p^2)\left(\delta^{ij} - \frac{p^i p^j}{\vec p^2}\right)$ to find $A(\vec p^2)$

take trace. contract w/ δ_{ij}: $\delta^{ij}\delta_{ij} = 3$

$2 = 3A(\vec p^2) + \underbrace{\vec p\, B(\vec p^2)}_{-A(\vec p^2)} \Rightarrow 2A(\vec p^2) \Rightarrow A(\vec p^2) = 1$.

So $\boxed{\sum_\lambda \epsilon^i(\vec p,\lambda)\, \epsilon^j(\vec p,\lambda) = \delta^{ij} - \frac{p^i p^j}{p^2}}$ → transverse projection operator. Now return to

$[\pi^i(x), A^j(y)] = \frac{i}{2}\int [d^3p] \sum_\lambda \epsilon^i(p,\lambda)\epsilon^j(p,\lambda) \left(e^{ip\cdot(x-y)} + e^{-ip(x-y)} \right)$.

$= i\int [d^3p] \left(\delta^{ij} - \frac{p^i p^j}{p^2}\right) e^{i\vec p\cdot(\vec x-\vec y)} = i\left(\delta^{ij} - \frac{\partial_x^i \partial_x^j}{\nabla^2}\right)\int [d^3p]\, e^{ip\cdot(\vec x-\vec y)}$

$\boxed{[\pi^i, A^j] = i\left(\delta^{ij} - \frac{\partial_x^i \partial_x^j}{\nabla_x^2}\right)\delta^3(x-y)}$ → as we guessed earlier.

Transverse projected δ-fn.

Egy mom tensor. $T^{\mu\nu} = -F^{\mu}{}_{\lambda} F^{\nu\lambda} + \frac{1}{4}\eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}$.

$E^i = -F^{0i}$ $\quad F^{ij} = \epsilon^{ijk} B^k$. $\Rightarrow \mathcal{H} = T^{00} = \frac{1}{2}(E^2 + \vec{B}^2)$

get H in terms of modes. $H = \frac{1}{2}\int d^3x\,(\vec{E}^2 + \vec{B}^2)$ ..

Electric egy $H_E = \frac{1}{2}\int d^3x\, \vec{E}^2 = \frac{1}{2}\int d^3x\, \vec{\pi}^2$.

$$H_E = \frac{1}{2}\int d^3x \int [d^3p] \frac{iE_p}{\sqrt{2E_p}} \int \frac{[d^3q]}{\sqrt{2E_q}} iE_q \sum_{\lambda\lambda'} \vec{\epsilon}(\vec{p},\lambda)\cdot\vec{\epsilon}(\vec{q},\lambda')$$
$$(a_\lambda(\vec{p})e^{-ip\cdot x} - a^+_\lambda(\vec{p})e^{ip\cdot x})(a_{\lambda'}(\vec{q})e^{-iq\cdot x} - a^+_{\lambda'}(\vec{q})e^{iq\cdot x}).$$

do the $\vec{x}$ integral.

$$H_E = \frac{1}{2}\int [d^3p] \frac{iE_p}{\sqrt{2E_p}} \frac{iE_q}{\sqrt{2E_q}} [d^3q] \sum_{\lambda\lambda'} \vec{\epsilon}(p,\lambda)\cdot\vec{\epsilon}(q,\lambda')$$
$$(2\pi)^3 [\,a_\lambda(\vec{p})\, a_{\lambda'}(\vec{q})\, \delta^3(\vec{p}+\vec{q}) e^{-2ip^0x^0} - a_\lambda(\vec{p}) a^+_{\lambda'}(\vec{q}) \delta^3(\vec{p}-\vec{q})$$
$$- a^+_\lambda(\vec{p})\, a_{\lambda'}(\vec{q})\, \delta^3(\vec{p}-\vec{q}) + a^+_\lambda(\vec{p}) a^+_{\lambda'}(\vec{q})\, \delta^3(p+q) e^{2ip^0x^0}$$

$$H_E = -\frac{1}{2}\int [d^3p] \frac{E_p}{2} \sum_{\lambda\lambda'} [\underbrace{\vec{\epsilon}(p,\lambda)\cdot\vec{\epsilon}(\vec{p},\lambda')}_{\delta_{\lambda\lambda'}} (a_\lambda(\vec{p}) a^+_{\lambda'}(\vec{p}) + a^+_\lambda(p) a_{\lambda'}(\vec{p}))$$
$$- \vec{\epsilon}(p,\lambda)\cdot\vec{\epsilon}(-p,\lambda')(a_\lambda(\vec{p}) a_{\lambda'}(-\vec{p})\, e^{-2ip^0x_0} + a^+_\lambda(\vec{p}) a^+_{\lambda'}(-\vec{p})\, e^{2ip^0x_0})$$

$$H_E = \frac{1}{4}\int [d^3p]\, E_p \sum_\lambda (a_\lambda(\vec{p}) a^+_\lambda(\vec{p}) + a^+_\lambda(p) a_\lambda(\vec{p}))$$
$$-\frac{1}{4}\int [d^3p]\, E_p \sum_{\lambda\lambda'} \vec{\epsilon}(p,\lambda)\cdot\vec{\epsilon}(-\vec{p},\lambda)(a_\lambda(\vec{p}) a_{\lambda'}(\vec{p})\, e^{-2ip^0x_0} + a^+_\lambda(p) a^+_{\lambda'}(-\vec{p})\, e^{2ip^0x_0})$$

if one takes VEV $\langle 0|H_E|0\rangle$ the second term vanishes

$$H_{mag} = \frac{1}{4}\int d^3\vec{x}\, F_{ij} F^{ij} \qquad F^{ij} = \partial^i A^j - \partial^j A^i = -\partial_i A^j + \partial_j A^i$$

$$F^{ij} = -i\int \frac{[d^3p]}{\sqrt{2E_p}} \sum_\lambda (p^i \epsilon^j(\vec{p},\lambda) - p^j \epsilon^i(\vec{p},\lambda))(a_\lambda(\vec{p})e^{-ip\cdot x} - a^+_\lambda(p) e^{ip\cdot x})$$

Thus: $$H_M = -\frac{1}{4}\int \frac{[d^3p][d^3q]}{\sqrt{2E_p 2E_q}} \sum_{\lambda\lambda'} (p^i \epsilon^j(\vec{p},\lambda) - p^j \epsilon^i(p,\lambda))(q^i \epsilon^j(q,\lambda) - q^j \epsilon^i(q,\lambda))$$
$$(a_\lambda(\vec{p})\, e^{-ip\cdot x} - a^+_\lambda(p)\, e^{ip\cdot x})(a_{\lambda'}(\vec{q})\, e^{-iq\cdot x} - a^+_{\lambda'}(\vec{q})\, e^{iq\cdot x}).$$

$$H_M = \frac{1}{4}\int [d^3\vec{p}]\,\frac{1}{2E_p}\sum_{\lambda\lambda'} 2\vec{p}^{\,2}\,\vec{E}(\vec{p},\lambda)\cdot\vec{E}(\vec{p},\lambda')\left(a_\lambda(p)a^\dagger_{\lambda'}(\vec{p}) + a^\dagger_\lambda(\vec{p})\,a_{\lambda'}(p)\right)$$

using $p\cdot\epsilon = 0$ & $q\cdot\epsilon(q,\lambda)=0$.

$$+\frac{1}{4}\int [d^3p]\frac{1}{2E_p}\sum_{\lambda\lambda'} 2p^2\,\vec{E}(p,\lambda)\vec{E}(-p,\lambda')\left(a_\lambda(\vec{p})\,a_{\lambda'}(-\vec{p})\,e^{-2ip^0x_0} + a^\dagger_\lambda(\vec{p})\,a^\dagger_{\lambda'}(-\vec{p})\,e^{2ip^0x_0}\right).$$

$H_M = (H_E)_{1st\ term} - (H_E)_{2nd\ term}$ ← Comparing we notice

Thus $\boxed{H = \frac{1}{2}\int [d^3p]\,E_p\sum_\lambda \left(a^\dagger_\lambda(p)\,a_\lambda(\vec{p}) + a_\lambda(p)\,a^\dagger_\lambda(p)\right)}$. check that

$[H, a^\dagger] = E_p\, a^\dagger$ & $[H, a] = -E_p\, a$ $\boxed{:H: = \int d^3p\; E_p\sum_\lambda a^\dagger_\lambda(p)\,a_\lambda(\vec{p})}$

$|0\rangle$ = vacuum. $a_\lambda(p)\,|0\rangle = 0$ Multiphoton states

$a^\dagger_{\lambda_n}(\vec{p}_n)\;\cdots\;a^\dagger_{\lambda_1}(p_1)\,|0\rangle$

Tomorrow spin of photon field; helicity $P^i = -\int d^3x\, \Pi^j F^{ij}$.

Momentum of photon field.

Can show $\boxed{:\vec{P}: = \int [d^3p]\;\vec{p}\;\sum_\lambda a^\dagger(\vec{p})\;a_\lambda(\vec{p})}$.

Tue 28 Aug. 8th Lecture

Spin of photon field $J^i = \frac{\epsilon^{ijk}}{2}\int d^3\vec{x}\; M^{0jk} = \frac{1}{2}\epsilon^{ijk}\int d^3x\,(x^jT^{0k} - x^kT^{0j})$.

$J^i = \cancel{\frac{1}{2}}\,\epsilon^{ijk}\int d^3\vec{x}\; x^j\,\dot{A}^l\,F^{kl}$ look at J^3

$T^{0k} = -F^{0\mu}F^k{}_\mu = -F^{0l}F^k{}_l = \dot{A}^l F^{kl}$.

$-E^l = F^{0l} = \dot{A}^l$

$$J^3 = \int d^3x\,(x^1F^{2l} - x^2F^{1l})\,\dot{A}^l$$
$$= -\int d^3x\,\dot{A}^l\left(x^1(\partial_2A^l - \partial^lA_2) - x^2(\partial_1A^l - \partial^lA_1)\right)$$

$$J^3 = \underbrace{-\int d^3x\,\dot{A}^l[x^1\partial_2 - x^2\partial_1]A^l}_{\text{orbital ang. mom.} \to \text{ignore hence forth}} + \int d^3x\,\dot{A}^l[x^1\partial_lA^2 - x^2\partial_lA^1]$$ simplify this before putting in Mode exp.

$$J^3_{spin} = \int d^3x\,\dot{A}^l[\partial_l(x^1A^2) - \delta^1_lA^2 - \partial_l(x^2A_1) + \delta^2_lA^1].$$
$$= \int d^3x\,\underbrace{\dot{A}^l\,\partial_l(x^1A^2 - x^2A^1)} - \int d^3x\,[\dot{A}^1A^2 - \dot{A}^2A^1].$$

divergence term ← $\partial_l\{\dot{A}^l(x^1A^2 - x^2A^1)\} - \vec{\nabla}\cdot\dot{\vec{A}}\,(x^1A^2 - x^2A^1)$. So. we get

$\vec{\nabla}\cdot\vec{E} = 0$

$\boxed{J^3 = \int d^3\vec{x}\,(\dot{A}^2A^1 - \dot{A}^1A^2)}$ nice & simple formula. Now we insert mode exp.

$$J^3 = \int d^3x \int \frac{[d^3p][d^3q]}{\sqrt{2E_p 2E_q}} (-iE_p) \sum_{\lambda\lambda'} \epsilon^2(\vec{p},\lambda)\epsilon^1(\vec{q},\lambda') .$$

$$\left[(a_\lambda(\vec{p}) e^{-ip\cdot x} - a^+_\lambda(\vec{p}) e^{ip\cdot x})(a_{\lambda'}(\vec{q}) e^{-iq\cdot x} + a^+_{\lambda'}(\vec{q}) e^{iq\cdot x}) - (1\leftrightarrow 2) \right]$$

Now need to do the x- integrations.

$$J^3 = \int \frac{[d^3p\, d^3q]}{\sqrt{2E_p 2E_q}} (-iE_p) \sum_{\lambda\lambda'} \epsilon^2(\vec{p},\lambda)\epsilon^1(\vec{q},\lambda)$$

$$(2\pi)^3 \left[a_\lambda(\vec{p}) a_{\lambda'}(\vec{q}) \delta^3(p+q) e^{-2ip^0x_0} + a_\lambda(\vec{p}) a^+_{\lambda'}(q) \delta^3(p-q) \right.$$

$$\left. - a^+_\lambda(\vec{p}) a_{\lambda'}(q) \delta^3(\vec{p}-\vec{q}) - a^+_\lambda(\vec{p}) a^+_{\lambda'}(\vec{q}) e^{2ip^0x_0} \delta^3(p+q) - (1\leftrightarrow 2) \right]$$

$$= -\frac{i}{2}\int [d^3p] \sum_{\lambda\lambda'} \left[\epsilon^2(\vec{p},\lambda)\epsilon^1(-\vec{p},\lambda') a_\lambda(\vec{p}) a_{\lambda'}(-\vec{p}) \cdot e^{-2ip^0x^0} \right. \quad \text{— (a)}$$

$$+ a_\lambda(p) a^+_{\lambda'}(\vec{p}) \epsilon^2(\vec{p},\lambda)\epsilon^1(\vec{p},\lambda') - a^+_\lambda(p) a_{\lambda'}(\vec{p}) \epsilon^2(\vec{p},\lambda)\epsilon^1(\vec{p},\lambda')$$

$$\left. - \underbrace{a^+_\lambda(\vec{p}) a^+_{\lambda'}(-\vec{p}) \epsilon^2(\vec{p},\lambda)\epsilon^1(-\vec{p},\lambda')}_{(d)} \right] - (1\leftrightarrow 2).$$

Consider the various terms

$$(a) = -\frac{i}{2}\int [d^3p] \sum_{\lambda\lambda'} \epsilon^2(\vec{p},\lambda)\cdot\epsilon^1(-p,\lambda') a_\lambda(\vec{p}) a_{\lambda'}(-\vec{p}) e^{-2ip^0x_0}$$ $1\leftrightarrow 2$ contrib

$$+\frac{i}{2}\int [d^3p] \sum_{\lambda\lambda'} \epsilon^1(p,\lambda)\epsilon^2(-\vec{p},\lambda') a_\lambda(\vec{p}) a_{\lambda'}(-\vec{p}) e^{-2ip^0x_0}$$ here send $\vec{p}\to-\vec{p}$ & $\lambda\leftrightarrow\lambda'$

then we find the two terms vanish when added.

So (a) $=0$ & (d) $=0$ so left w/ the middle terms.

$$J^3 = -\frac{i}{2}\int [d^3p] \sum_{\lambda\lambda'} \epsilon^2(\vec{p},\lambda)\epsilon^1(\vec{p},\lambda') \left(a_\lambda(p) a^+_{\lambda'}(\vec{p}) - a^+_\lambda(\vec{p}) a_{\lambda'}(\vec{p}) \right)$$

$$+\frac{i}{2}\int [d^3p] \sum_{\lambda\lambda'} \epsilon^1(\vec{p},\lambda')\epsilon^2(\vec{p},\lambda) \left(a_{\lambda'}(p) a^+_{\lambda}(\vec{p}) - a^+_{\lambda'}(\vec{p}) a_{\lambda}(p) \right)$$ 1-2 each term

upon normal ordering the ~~two~~ two terms are the same.

$$\Rightarrow \boxed{J^3 = i\int [d^3p] \sum_{\lambda\lambda'} \epsilon^2(\vec{p},\lambda)\epsilon^1(\vec{p},\lambda') \left(a^+_\lambda(\vec{p}) a_{\lambda'}(p) - a^+_{\lambda'}(\vec{p}) a_\lambda(p) \right)}$$

Now consider 1-particle states call $\lambda\to\rho$ ρ = polarization index $\rho=1,2$

$$J_3 a^+_\rho(\vec{p})|0\rangle = [J_3, a^+_\rho(\vec{p})]|0\rangle$$

$$= i\int [d^3q] \sum_{\lambda\lambda'} \epsilon^2(\vec{q},\lambda)\epsilon^1(\vec{q},\lambda') \Big(a^+_\lambda(\vec{q}) \overbrace{[a_{\lambda'}(\vec{q}), a^+_\rho(\vec{p})]}^{\delta_{\lambda'\rho}(2\pi)^3\delta^3(p-q)}$$

$$- a^+_{\lambda'}(\vec{q}) \underbrace{[a_\lambda(\vec{q}), a^+_\rho(\vec{p})]}_{\delta_{\lambda\rho}\delta^3(p-q)(2\pi)^3} \Big)|0\rangle.$$

can make these commutators as extra term annih vacuum

$$= i\left[\sum_\lambda \epsilon^2(p,\lambda)\epsilon^1(\vec{p},\rho) a^+_\lambda(p) - \sum_{\lambda'} \epsilon^2(\vec{p},\rho)\epsilon^1(\vec{p},\lambda') a^+_{\lambda'}(\vec{p})\right]|0\rangle$$

$$= i\sum_\lambda \left[\epsilon^2(\vec{p},\lambda)\epsilon^1(\vec{p},\rho) a^+_\lambda(\vec{p}) - \epsilon^2(\vec{p},\rho)\epsilon^1(\vec{p},\lambda) a^+_\lambda(\vec{p}) \right]|0\rangle$$

Spin of 1-photon states (contd) Tue 28 Aug. Assume photon moving in $\hat{k}=\hat{z}$ direction.

$$J_3^{spin}\, a_\rho^\dagger(\vec{p})|0\rangle = i\,\epsilon^1(\vec{p},\rho)\sum_\lambda \epsilon^2(\vec{p},\lambda)\, a_\lambda^\dagger(\vec{p})|0\rangle - i\epsilon^2(\vec{p},\rho)\sum_\lambda \epsilon^1(\vec{p},\lambda)\, a_\lambda^\dagger(\vec{p})|0\rangle$$

write out terms

$$= i a_1^\dagger(\vec{p})\left[\epsilon^1(\vec{p},\rho)\,\epsilon^2(\vec{p},1) - \epsilon^2(\vec{p},\rho)\,\epsilon^1(\vec{p},1)\right]|0\rangle \leftarrow \text{non zero only if } \rho=2$$

$$+ i a_2^\dagger(\vec{p})\left[\epsilon^1(\vec{p},\rho)\,\epsilon^2(\vec{p},2) - \epsilon^2(\vec{p},\rho)\,\epsilon^1(\vec{p},2)\right]|0\rangle \leftarrow \text{non zero only if } \rho=1.$$

$$= i a_1^\dagger(\vec{p})\,\delta_{\rho 2}\underbrace{\left(\epsilon^1(\vec{p},2)\,\epsilon^2(\vec{p},1) - \epsilon^2(\vec{p},2)\,\epsilon^1(\vec{p},1)\right)}_{-1}|0\rangle$$

$$+ i a_2^\dagger(\vec{p})\,\delta_{\rho 1}\underbrace{\left(\epsilon^1(p,1)\,\epsilon^2(p,2) - \epsilon^2(p,1)\,\epsilon^1(p,2)\right)}_{+1}|0\rangle.$$

recall:

$\vec{\epsilon}(k,1)\times\vec{\epsilon}(k,2) = \hat{k}=\hat{z}$ write in components.

$$\epsilon^1(k,1)\,\epsilon^2(k,2) - \epsilon^2(k,1)\,\epsilon^1(k,2) = 1.$$

suppose propagation is in the 3rd direction.

So $J_3\, a_\rho^\dagger(\vec{p})|0\rangle = -i a_1^\dagger(\vec{p})\,\delta_{\rho 2}|0\rangle + i a_2^\dagger(\vec{p})\,\delta_{\rho 1}|0\rangle.$

In particular $J_3\, a_1^\dagger(\vec{p})|0\rangle = i a_2^\dagger(\vec{p})|0\rangle$

and $J_3\, a_2^\dagger(\vec{p})|0\rangle = -i a_1^\dagger(\vec{p})|0\rangle.$

So neither of the 1-photon states has defining J_3^{spin}.
but can take linear combs to get eigenstates → left & right circularly polarized photons.

$$J_3\left(\frac{a_1^\dagger(\vec{p}) + i a_2^\dagger(\vec{p})}{\sqrt{2}}\right)|0\rangle = \left(\frac{a_1^\dagger(\vec{p}) + i a_2^\dagger(\vec{p})}{\sqrt{2}}\right)|0\rangle \quad \text{and}$$

$$J_3\left(\frac{a_1^\dagger(\vec{p}) - i a_2^\dagger(\vec{p})}{\sqrt{2}}\right)|0\rangle = -\left(\frac{a_1^\dagger(\vec{p}) - i a_2^\dagger(\vec{p})}{\sqrt{2}}\right)|0\rangle.$$

So define right circ. polarized $a_R^\dagger(\vec{p}) = \frac{1}{\sqrt{2}}\left(a_1^\dagger(\vec{p}) + i a_2^\dagger(\vec{p})\right).$

and $a_L^\dagger(\vec{p}) = \frac{1}{\sqrt{2}}\left(a_1^\dagger(\vec{p}) - i a_2^\dagger(\vec{p})\right)$. → Left circ pol.

$J_3\, a_R^\dagger(\vec{p})|0\rangle = a_R^\dagger(\vec{p})|0\rangle$ and $J_3\, a_L^\dagger(\vec{p})|0\rangle = -a_L^\dagger(\vec{p})|0\rangle.$

In Proca field (massive vector field) get $J_3 = +1, -1, 0$ all 3 are all dynamical
A^0 still non dynamical but A^1, A^2, A^3 are all dynamical

Feynman propagator for Photon field

put in mode expansion.

$$D_F(x-y)_{ij} = \langle 0|T A_i(x) A_j(y)|0\rangle$$

$$= \theta(x^0-y^0)\langle 0|A_i(x) A_j(y)|0\rangle + \theta(y^0-x^0)\langle 0|A_j(y) A_i(x)|0\rangle$$

$$= \theta(x^0-y^0)\langle 0|\int\frac{d^3q}{\sqrt{2E_q}}\sum_{\lambda'} a_{\lambda'}(\vec{q})\,\epsilon(q,\lambda')\,e^{-iq\cdot x}\int\frac{d^3p}{\sqrt{2E_p}}\sum_\lambda a_\lambda^\dagger(\vec{p})\,\epsilon(p,\lambda)\,e^{ip\cdot y}|0\rangle.$$

$+ \cdots$

$$D_F(x-y) = \theta(x^0-y^0)\int \frac{[d^3p]}{\sqrt{2E_p}} \sum_\lambda \epsilon_i(p,\lambda)\, \epsilon_j(p,\lambda)\, e^{-ip\cdot(x-y)}$$

$$+\ \theta(y^0-x^0)\int \frac{[d^3p]}{\sqrt{2E_p}} \sum_\lambda \epsilon_j(p,\lambda)\,\epsilon_i(p,\lambda)\, e^{ip\cdot(x-y)}$$

$$= \int \frac{[d^3p]}{2E_p} \underbrace{\sum_\lambda \epsilon_i(p,\lambda)\,\epsilon_j(p,\lambda)}_{\delta_{ij} - \frac{p_i p_j}{\vec{p}^2}} \left[\theta(x^0-y^0)e^{-ip\cdot(x-y)} + \theta(y^0-x^0)\, e^{ip\cdot(x-y)}\right]$$

$$= \int [d^4p] \frac{i}{p^2+i\epsilon} e^{-ip\cdot(x-y)} \sum_\lambda \epsilon_i(\vec{p},\lambda)\,\epsilon_j(p,\lambda) = D_F(x-y)_{ij}$$

This is in Radiation gauge. Sometimes useful to have a covariant expression ~~(in~~

$$D_F(x-y)_{\mu\nu} = \langle 0|\,T A_\mu(x) A_\nu(y)\,|0\rangle = \int [d^4p] \frac{i e^{-ip\cdot(x-y)}}{p^2+i\epsilon} \sum_\lambda \epsilon_\mu(p,\lambda)\,\epsilon_\nu(\vec{p},\lambda)$$

This reduces to the expr in radiation gauge.

Want to generalize our expr to covariant form.

$\epsilon_\mu(\vec{k},1)$; $\epsilon_\mu(\vec{k},2)$ $\qquad \epsilon_\mu = (0, \vec{\epsilon})$

introduce $\eta^\mu = (1, \vec{0})$ → in radiation gauge (time like)

with $\eta^\mu \eta_\mu = 1$. time like unit vector.

Need the 3rd polarization vector

$$\hat{k}^\mu = \frac{k^\mu - \eta^\mu (k\cdot\eta)}{\sqrt{(k\cdot\eta)^2 - k^2}}$$

check $\hat{k}^i$ in Rad. gauge

$$\hat{k}^i = \frac{k^i}{\sqrt{(k^0)^2 - k^2}} = \frac{k^i}{\sqrt{\vec{k}^2}}.$$

and $\hat{k}^0 = \dfrac{k^0 - k^0}{|\vec{k}|} = 0$. reduces to what we had in rad gauge

also calc $\quad \hat{k}^\mu \hat{k}_\mu = \dfrac{(k^\mu - \eta^\mu(k\cdot\eta))(k_\mu - \eta_\mu(k\cdot\eta))}{(\eta\cdot k)^2 - k^2} = \dfrac{k^2 - 2(k\cdot\eta)^2 + (k\cdot\eta)^2}{(\eta\cdot k)^2 - k^2}$

$\hat{k}^\mu \hat{k}_\mu = -1$ $\qquad$ so $\hat{k}^\mu$ is space like

to gauge fix in Lorentz cov. gauge $\quad -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$

you gauge fix $\mathcal{L}$ this way for

4 o.n. vectors $\quad \epsilon_\mu(\vec{k},1),\ \epsilon_\mu(k,2),\ \eta_\mu,\ \hat{k}_\mu$

η_μ ↓ time like $\qquad \hat{k}_\mu$ ↓ space like

Now to get the propagator in cov. gauge we need to calc pol. sum. (44)

$$\sum_\lambda \epsilon_\mu(\vec{k},\lambda)\,\epsilon_\nu(\vec{k},\lambda) = A(\vec{k}^2, k\cdot n)\eta_{\mu\nu} + B(\vec{k}^2, k\cdot n)\hat{k}_\mu \hat{k}_\nu + C(\vec{k}^2, k\cdot n)\,\eta_\mu \eta_\nu$$

$$+ D(\vec{k}^2, k\cdot n)\,(\eta_\mu k_\nu + \eta_\nu k_\mu). \quad \nearrow D_{\mu\nu}(k)$$

$\vec{k}^2$, $k\cdot n$ are the scalars available. $k^2 \neq 0$ here (not on shell)

Specialize to rad gauge. take $\mu=\nu=0$, $0 = A + C$.

$\mu=0,\ \nu=i \quad 0 = D$, $\quad \mu=i, \nu=j \quad \delta_{ij} - \dfrac{k_i k_j}{|\vec{k}|^2} = -A\delta_{ij} + \dfrac{B k_i k_j}{|\vec{k}|^2}$

So $A=-1,\ B=-1,\ C=1,\ D=0$.

So $\sum_\lambda \epsilon_\mu(k,\lambda)\,\epsilon_\nu(k,\lambda) = -\eta_{\mu\nu} - \hat{k}_\mu\hat{k}_\nu + \eta_\mu\eta_\nu$.

So $D_F(x-y)_{\mu\nu} = \displaystyle\int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2+i\epsilon}\,e^{-ip\cdot(x-y)}\,(-\eta_{\mu\nu} + n_\mu n_\nu - \hat{p}_\mu\hat{p}_\nu)$.

Substitute $n_\mu n_\nu - \hat{p}_\mu\hat{p}_\nu = n_\mu n_\nu - \dfrac{(p_\mu - n_\mu(k\cdot n))(p_\nu - n_\nu(k\cdot n))}{(k\cdot n)^2 - k^2}$.

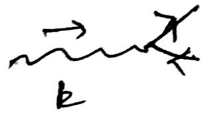
$$D_F(x-y)_{\mu\nu} = \int [d^4p]\,\frac{i}{p^2+i\epsilon}\,e^{-ip\cdot(x-y)}\left[\frac{-p^2 n_\mu n_\nu}{(p\cdot n)^2 - p^2} - \overbrace{\cancel{\frac{p_\mu p_\nu}{(p\cdot n)^2 - p^2}}}^{\to 0\ \mathrm{WT}} + \overbrace{\cancel{\frac{(p\cdot n)(p_\mu n_\nu + p_\nu n_\mu)}{(p\cdot n)^2 - p^2}}}^{\to 0\ \mathrm{WT}}\right]$$

Now to couple photon to current $\to$ at linearized level gauge invariance.

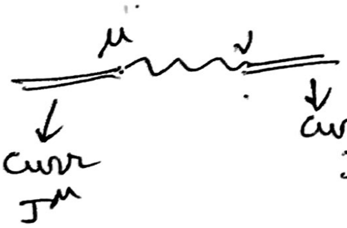
$$\int -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,d^4x + \int d^4x\,J^\mu A_\mu$$

req. $A_\mu' \to A_\mu + \partial_\mu\lambda \quad$ So $\int d^4x\,J^\mu\partial_\mu\lambda = 0$

So need $\partial^\mu J_\mu = 0$. This is tree level pf of Ward-Takahashi (WT) identity. More gen.



$J^\mu(k)\,k_\mu = 0$.



propagator sandwiched betw conserved currents

Since $J^\nu k_\nu = 0$ & $J^\mu k_\mu = 0$ by W-T

we can ignore ~~some~~ terms from $D^F_{\mu\nu}(x-y)$.

$\hookrightarrow$ last 2 terms.

Consider

$$-\int [d^4p]\,\frac{i}{\ }\,\eta_\mu\eta_\nu\,\frac{e^{-ip\cdot(x-y)}}{(p\cdot n)^2 - p^2} \underset{\substack{\uparrow \\ \text{Rad Gauge} \\ \eta_\mu = \eta_{\mu 0} = (1,\vec{0})}}{=} -i\,\eta_{\mu 0}\eta_{\nu 0}\int\frac{dp^0}{2\pi}\int[d^3p]\,\frac{e^{-ip^0(x-y)^0}\,e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}}{\vec{p}^{\,2}}$$

$\delta(x^0-y^0)$

$$= -i\,\eta_{\mu 0}\eta_{\nu 0}\,\delta(x^0-y^0)\underbrace{\int[d^3p]\,\frac{e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}}{\vec{p}^{\,2}}}_{G(\vec{x}-\vec{y})}$$

$\nabla_x^2 G(x-y) = -\delta^3(\vec{x}-\vec{y})$

$G(\vec{x}-\vec{y}) = \dfrac{1}{4\pi}\dfrac{1}{|\vec{x}-\vec{y}|}$.

So $D_F(x-y)_{\mu\nu} = \int [d^4p] \dfrac{-i\eta_{\mu\nu}}{p^2+i\epsilon} e^{-ip\cdot(x-y)} - i\eta_{\mu 0}\eta_{\nu 0} \dfrac{\delta(x^0-y^0)}{4\pi |x-y|}$.

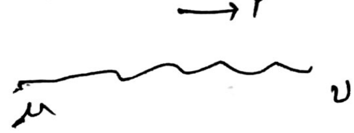
In rad gauge ignored matter & coulomb

$\underbrace{\qquad\qquad}_{\text{Coulomb interaction}}$

In rad gauge have ignored Coulomb part so must add it and that cancels the last term

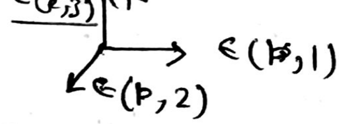
So $D_F(x-y)_{\mu\nu} = \int [d^4p]\, D_F(p)_{\mu\nu}\, e^{-ip\cdot(x-y)}$

when $D_F(p)_{\mu\nu} = \dfrac{-i\eta_{\mu\nu}}{p^2+i\epsilon}$



This is propagator in $\xi = 1$ covariant gauge. $e^{i\omega t}$

Next:



Interacting Fields and Feynman diagrams: (Wednesday 29 Aug)

Perturbative expansion of correlation fns

$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$ $\qquad$ λ small $\Rightarrow$ perturbation theory

Want $\langle\Omega| T\,\phi(x)\,\phi(y)|\Omega\rangle$ $\qquad$ $|\Omega\rangle$ = interacting vacuum

$|\Omega\rangle \xrightarrow{\lambda\to 0} |0\rangle$ free vacuum.

take $H = H_0 + H_{int}$ $\qquad$ $H_{int} = \frac{\lambda}{4!}\int d^3\vec{x}\,\phi^4$. $\qquad$ $H_0 = \frac{1}{2}(\dot\phi^2 + (\nabla\phi)^2 + m^2\phi^2)$.

Go back to free fields: put $t=0$ (Schrodinger pict)

$\phi(x) = \int \frac{[d^3p]}{\sqrt{2E_p}} [\, a_p e^{i\vec{p}\cdot\vec{x}} + a_p^+ e^{-i\vec{p}\cdot\vec{x}}\,]$ $\qquad$ Now go to Heisenberg pict $\boxed{e^{iH_0 t}(\;)e^{-iH_0 t}}$

schrodinger pict time indep

In Heisenberg pict:

$\phi(x,t) = \int \frac{[d^3p]}{\sqrt{2E_p}} [\,(e^{iH_0t} a_p e^{-iH_0 t}) e^{i\vec{p}\cdot\vec{x}} + (e^{iH_0t} a_p^+ e^{-iH_0t}) e^{-i\vec{p}\cdot\vec{x}}\,]$

use Heisenberg EoM: $\quad i\frac{\partial}{\partial t} a_p(t) = [a_p(t), H_0] = e^{iHt}\underbrace{[a_p, H_0]}_{E_p a_p} e^{-iHt}$

So $\quad i\partial_t a_p(t) = E_p a_p(t) \Rightarrow a_p(t) = a_p e^{-iE_pt}$

and $a_p^+(t) = a_p^+ e^{iE_pt}$

Hence forth use $a_p(t)\; a_p^+(t)$ Heisenberg field operators.

Go back to interacting theory

at $t=t_0$ suppose interactions are negligible. then $\varphi(x,t_0)$ can be given a free field expansion

t_0: say a microsecond before the "particles" are appreciably near each other

$\varphi(x,t_0) = \int \frac{(d^3p)}{\sqrt{2E_p}} (\, a_p(t_0) e^{i\vec{p}\cdot\vec{x}} + a_p^+(t_0) e^{-i\vec{p}\cdot\vec{x}}\,)$

Now we go forward to time $t > t_0$. 46

$$\varphi(\vec{x}, t) = e^{iH(t-t_0)}\, \varphi(x, t_0)\, e^{-iH(t-t_0)}.$$

If $\lambda = 0$ have $H = H_0$. We will need the "interaction picture"

$$\varphi(x,t)\Big|_{\lambda=0} = e^{iH_0(t-t_0)}\, \varphi(\vec{x}, t_0)\, e^{-iH_0(t-t_0)} \equiv \varphi_I(\vec{x}, t).$$

Interaction pict field is the Heisenberg picture field wrto the free hamiltonian H_0.

$$\varphi_I(\vec{x}, t) = e^{iH_0(t-t_0)} \int \frac{d^3p}{\sqrt{2E_p}} \left(a_p(t_0)\, e^{i\vec{p}\cdot\vec{x}} + a_p^\dagger(t_0)\, e^{-i\vec{p}\cdot\vec{x}} \right) e^{-iH_0(t-t_0)}.$$

$$= \int \frac{[d^3p]}{\sqrt{2E_p}} \left(a_p(t_0)\, e^{-iE_p(t-t_0) + i\vec{p}\cdot\vec{x}} + a_p^\dagger(t_0)\, e^{iE_p(t-t_0) - i\vec{p}\cdot\vec{x}} \right).$$

using the free field evolution of the free creation-annihilation ops

If we introduce $x^0 = t - t_0$ then, we have a 4-vector notation.

$$\varphi_I(\vec{x}, t) = \int \frac{[d^3p]}{\sqrt{2E_p}} \left[a_p(t_0)\, e^{-ip\cdot x} + a_p^\dagger(t_0)\, e^{ip\cdot x} \right].$$

Here $a_p(t) = e^{iH_0(t-t_0)}\, a_p(t_0)\, e^{-iH_0(t-t_0)} = a_p(t_0)\, e^{-iE_p(t-t_0)}$.

$$\varphi(x,t) = e^{iH(t-t_0)}\, e^{-iH_0(t-t_0)}\, \varphi_I(x,t)\, e^{iH_0(t-t_0)}\, e^{-iH(t-t_0)}.$$

where by defn. $\varphi_I(x,t) = e^{iH_0(t-t_0)}\, \varphi(x, t_0)\, e^{-iH_0(t-t_0)}$.

Now introduce

$$U(t, t_0) = e^{iH_0(t-t_0)}\, e^{-iH(t-t_0)}$$

$$U^\dagger(t, t_0) = e^{iH(t, t_0)}\, e^{-iH_0(t-t_0)}.$$

Want a more useful expr for $U(t, t_0)$:

$$i\frac{\partial}{\partial t} U(t, t_0) = e^{iH(t-t_0)} \underbrace{(-H_0 + H)}_{H_{int}} e^{-iH_0(t-t_0)}$$

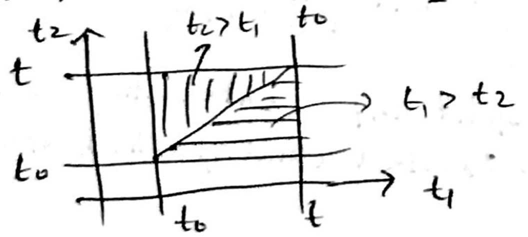
$$= \underbrace{e^{iH_0(t-t_0)}\, H_{int}\, e^{-iH_0(t-t_0)}}_{H_I \to \text{ perturbing hamiltonian in interaction picture}} U(t, t_0).$$

So $\boxed{i\dfrac{\partial U}{\partial t}(t, t_0) = H_I(t)\, U(t, t_0) \text{ and w/ IC. } U(t_0, t_0) = I.}$

Solution of this eqn is a time ordered exponential.

$\boxed{U(t, t_0) = T \exp -i\int_{t_0}^{t} dt'\, H_I(t')}$ Dyson formula. Here

$$U(t, t_0) = I - i\int_{t_0}^{t} dt'\, H_I(t') + \frac{(-i)^2}{2!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, T(H_I(t_1)\, H_I(t_2)) + \cdots$$



Let us check that this solves the differential equation

The quadratic term is:

$$\frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \; H_I(t_1) H_I(t_2) + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \; H_I(t_2) H_I(t_1)$$

$t_1 > t_2$ $\qquad\qquad$ $t_2 > t_1$

switch $t_1 \leftrightarrow t_2$ dummy var of integration in 2nd term

$$= \quad '' \quad + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \; H_I(t_1) H_I(t_2)$$

so we can add the two terms & the $\frac{1}{2!}$ cancels.

More generally get at n^{th} order (reversing $1\,2\,3 \ldots n \leftrightarrow n\,(n-1) \ldots 3\,2\,1$ notation.)

$$(-i)^n \int_{t_0}^{t} dt_n \int_{t_0}^{t_n} dt_{n-1} \cdots \int_{t_0}^{t_3} dt_2 \int_{t_0}^{t_2} dt_1 \; H_I(t_n) \cdots H_I(t_1).$$

then $n!$ will cancel as the n^{dim} cube is broken into $n!$ simplices corresponding to the $n!$ possible orderings of the times

Thus Dyson proposes

$$U(t,t_0) = I - i\int_{t_0}^{t} dt' \, H_I(t') - \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \, H_I(t_2) H_I(t_1) + \cdots$$

$$\Rightarrow i\partial_t U(t,t_0) = H_I(t) - i\int_{t_0}^{t} dt_1 \, H_I(t) H_I(t_1) + \cdots$$

$$= H_I(t)\left[I - i\int_{t_0}^{t} H_I(t)\, dt + \cdots \right]$$

$$= H_I(t) \, U(t,t_0) \quad \text{as desired.}$$

Summary

$$\varphi(\vec{x},t) = U^+(t,t_0)\, \varphi_I(\vec{x},t)\, U(t,t_0).$$

where $U(t,t_0) = T\exp -i\int_{t_0}^{t} H_I(t')\,dt'$

useful properties: $U(t_1 t_2) U(t_2 t_3) = U(t_1\, t_3)$ prop from t_3 to t_2 to t_1

$U(t_1\, t_3)\, U^+(t_2\, t_3) = U(t_1, t_2)$.

The complications are now in $U(t,t_0)$.

AIM: use U to express interacting vacuum in terms of free vacuum.

Strategy: $|0\rangle$ is g.s of H_0 $\quad H_0|0\rangle = 0$. where $|n\rangle$ is a complete set of eigenstates of H.

look @ $\quad e^{-iHT}|0\rangle = \sum_n e^{-iHT}|n\rangle\langle n|0\rangle$

$$= e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{\tilde{n}\neq\Omega} e^{-iE_{\tilde{n}}T}|\tilde{n}\rangle\langle\tilde{n}|0\rangle.$$

where $\langle\Omega|H|\Omega\rangle = E_0 > 0 \rightarrow$ as $\lambda > 0$ $\quad$ split into interacting vacuum & excited (multiparticle) states.

$H|\tilde{n}\rangle = E_{\tilde{n}}|n\rangle$

want to take $T \to \infty(1-i\epsilon)$ so that the excited state contributions are damped relative to the interacting vac $|\Omega\rangle$.

$$\lim_{T\to\infty(1-i\epsilon)} e^{-iHT}|0\rangle = \lim_{T\to\infty(1-i\epsilon)} e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle.$$

This is achieved by giving T a small negative imaginary part later take $\epsilon \to 0^+$.

So $$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{e^{-iHT}|0\rangle}{\langle\Omega|0\rangle e^{-iE_0T}}.$$

want to write in terms of U.

use $H_0|0\rangle = 0$

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{e^{iHt_0} e^{-iH(T+t_0)} e^{iH_0(t_0+T)}|0\rangle}{\langle\Omega|0\rangle \exp(-iE_0(T+\cancel{t_0}))}$$

$U^\dagger(-T,t_0) = e^{-iH(T+t_0)} e^{iH_0(T+t_0)}$ by defn.

Now mult both sides by e^{-iHt_0} get $e^{-iE_0t_0}$ on LHS & bring it down to RHS then

$$\boxed{|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{U^\dagger(-T,t_0)|0\rangle}{\langle\Omega|0\rangle e^{-iE_0(T+t_0)}}}$$

t_0 small arbitrary.

Now do the same for the interacting bra vacuum.

$$\langle 0|e^{iHT} = \sum_n \langle 0|n\rangle\langle n|e^{iHT} = \sum \langle 0|\Omega\rangle\langle\Omega|e^{iE_0T} + \sum_{\tilde{n}\neq\Omega} \langle 0|\tilde{n}\rangle\langle\tilde{n}|e^{iE_{\tilde{n}}T}$$

we get: $$\lim_{T\to-\infty(1-i\epsilon)} \langle 0|e^{iHT} = \langle 0|\Omega\rangle\langle\Omega|e^{iE_0T}$$

proceeding as before: $$\boxed{\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U(T,t_0)}{\langle 0|\Omega\rangle e^{-iE_0(T-t_0)}}}$$

Now return to calculating the 2-pt vac. time ordered correlation fn

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \langle\Omega|\varphi(x)\varphi(y)|\Omega\rangle.$$

$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U(T,t_0)\, U^\dagger(x^0,t^0)\varphi_I(x)U(x^0,t^0)\, U^\dagger(y^0,t^0)\varphi_I(y)U(y^0,t^0)}{\langle 0|\Omega\rangle e^{-iE_0(T-t_0)}} \frac{U(t_0,-T)|0\rangle}{\langle\Omega|0\rangle e^{-iE_0(T+t_0)}}$$

$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U(T,x_0)\varphi_I(x)\underline{U(x^0,y^0)}\varphi_I(y)\,\underline{U(y^0,-T)}|0\rangle}{|\langle 0|\Omega\rangle|^2 e^{-2iE_0T}}$$

Now need to deal w/ denominator.

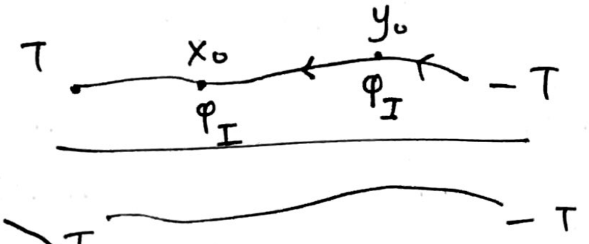
$$1 = \langle\Omega|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U(T,t_0)U(t_0,-T)|0\rangle}{|\langle 0|\Omega\rangle|^2 e^{-2iE_0T}} = \lim \frac{\langle 0|U(T,-T)|0\rangle}{|\langle 0|\Omega\rangle|^2 e^{-2iE_0T}}$$

φ_I = Interaction picture field operator

Thus

$$\langle\Omega|T\varphi(x)\varphi(y)|\Omega\rangle = \lim_{T\to\infty} \frac{\langle 0|U(T,x^0)\varphi_I(x)U(x^0,y^0)\,\varphi_I(y)\,U(y^0,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle}.$$

$=$



(have in mind the case $x^0 > y^0$)

pictorially evolving from early time $-T$ to late time T w/ 2 interactions in between & then take $T\to\infty$.

<u>So</u> $$\langle\Omega|T\varphi(x)\varphi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\,\varphi_I(x)\,\varphi_I(y)\,e^{-i\int_{-T}^{T}dt\,H_I(t)}|0\rangle}{\langle 0|T\,e^{-i\int_{-T}^{T}H_I(t)\,dt}|0\rangle}$$

The time ordering allows us to write this in a simple way. Now on all fields will be in the interaction picture so we will denote $\varphi_I(x) \equiv \varphi(x)$. Can generalize this formula to multi-point correlations & also to Dirac & Maxwell flds.

We need to calculate (upon expanding the time ordered exponential) vev (w.r.to free $|0\rangle$) of time-ordered products of interaction pict. operators.

<u>Wick's theorem</u>. for $\langle 0|T\varphi(x_1)\cdots\varphi(x_n)|0\rangle$.

$$\varphi(x) = \int \frac{[d^3p]}{\sqrt{2E_p}}\Big[\underbrace{a_p e^{-ip\cdot x}}_{\varphi^+(x)} + \underbrace{a_p^\dagger e^{ip\cdot x}}_{\varphi^-(x)}\Big]$$

positive & -ve frequency parts

$$\varphi^+(x)|0\rangle = 0 \quad \text{and} \quad \langle 0|\varphi^-(x) = 0.$$

<u>Now if $x^0 > y^0$</u> $\;T\varphi(x)\varphi(y) = \varphi(x)\varphi(y) = (\varphi^+(x)+\varphi^-(x))(\varphi^+(y)+\varphi^-(y))$

$$T(\varphi(x)\varphi(y)) = \varphi^+(x)\varphi^+(y) + \varphi^-(y)\varphi^+(x) + \varphi^-(x)\varphi^+(y) + \varphi^-(x)\varphi^-(y) + [\varphi^+(x),\varphi^-(y)].$$

These terms are normal ordered. $\quad :a_k\, a^\dagger_{k'}: \;=\; a^\dagger_{k'}\, a_k.$

Thus for $x^0>y^0$: $\quad \varphi(x)\varphi(y) = :\varphi(x)\varphi(y): + [\varphi^+(x),\varphi^-(y)]$.

Similarly for $y^0 > x^0$: $\quad \varphi(y)\varphi(x) = :\varphi(y)\varphi(x): + [\varphi^+(y),\varphi^-(x)]$

Thus: $$\langle 0|T\varphi(x)\varphi(y)|0\rangle = \theta(x^0-y^0)\{:\varphi(x)\varphi(y): + [\varphi^+(x),\varphi^-(y)]\}$$
$$+\,\theta(y^0-x^0)\{:\varphi(y)\varphi(x): + [\varphi^+(y),\varphi^-(x)]\}$$

But then $\theta(x^0-y^0)\ :\varphi(x)\varphi(y):\ +\ \theta(y^0-x^0):\varphi(y)\,\varphi(x):\ =\ :\varphi(x)\,\varphi(y):$

Thus $T\varphi(x)\varphi(y) = \ :\varphi(x)\varphi(y):\ + \theta(x^0-y^0)\,[\varphi^+(x), \varphi^-(y)] + \theta(y^0-x^0)\,[\varphi^+(y), \varphi^-(x)]$.

what is this ?

Recall: $D_F(x-y) = \theta(x^0-y^0)\,\langle 0|\varphi(x)\varphi(y)|0\rangle + \theta(y^0-x^0)\,\langle 0|\,\varphi(y)\varphi(x)\,|0\rangle$

$= \theta(x^0-y^0)\langle 0|\,\varphi^+(x)\,\varphi^-(y)\,|0\rangle +$ "

$= \theta(x^0-y^0)\,\langle 0|\,\underbrace{[\varphi^+(x), \varphi^-(y)]}_{\text{c-number}}\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,[\varphi^+(y), \varphi^-(y)]\,|0\rangle$

$= \theta(x^0-y^0)\ [\varphi^+(x), \varphi^-(y)] + \theta(y^0-x^0)\ [\varphi^+(y), \varphi^-(x)]$.

Thus we can write Wick's thm for 2 pt fn:

$$T(\varphi(x)\varphi(y)) = \ :\varphi(x)\varphi(y):\ + D_F(x-y)$$
$$= \ :\varphi(x)\,\varphi(y):\ + \overbrace{\varphi(x)\varphi(y)}$$

"contraction".

Moregenerally: $T(\varphi(x_1)\cdots\varphi(x_n)) = \ :\varphi(x_1)\cdots\varphi(x_n):\ + :\left\{\begin{matrix}\text{all possible}\\ \text{contractions.}\end{matrix}\right\}:$

Eg $T(\varphi_1\,\varphi_2\,\varphi_3\,\varphi_4) = \ :\varphi_1\varphi_2\varphi_3\varphi_4:\ + :\overbrace{1\,2}\,3\,4: + :\overbrace{1\,2\,3}\,4: + :\overbrace{1\,2\,3\,4}:$

$+ :1\,\overbrace{2\,3}\,4: + :1\,2\,\overbrace{3\,4}: + :\overbrace{1\,2}\,\overbrace{3\,4}: + :\overbrace{1\,2\,3}\,\underbrace{2\,3\,4}: + :\overbrace{1\,\underbrace{2\,3}\,4}:$

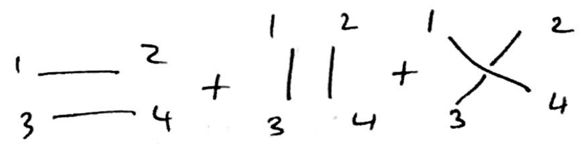
Where $:\overbrace{1\,2\,3}\,4: \ = D_F(x_1-x_3):\varphi(x_2)\,\varphi(x_4):$ etc.

Now suppose we want to calc a $\langle 0|T\cdots|0\rangle$ all the normal ordered terms w/ no uncontracted fields vanishes. leaving a sum over all possible complete contractions.

$$\langle 0|\,T\,\varphi_1\,\varphi_2\,\varphi_3\,\varphi_4\,|0\rangle = D_F(x_1-x_2)\,D(x_3-x_4)$$
$$+ D_F(x_1-x_3)\,D_F(x_2-x_4)$$
$$+ D_F(x_1-x_4)\,D_F(x_2-x_3).$$

Denote: $\underset{i\qquad j}{\longrightarrow} = D_F(x_i-x_j)$

Thus diagrammatically,

$\langle 0|\,T\varphi(x_1)\,\varphi(x_2)\,\varphi(x_3)\varphi(x_4)|0\rangle =$ 

Thus the space-time Feynman diagram interpretation of the time-ordered correlation fns.

Next: Feynman diag for φ^4 theory & QED.

Two point Function Thu 30 August 2018

$\langle\Omega|\,T\varphi(x)\varphi(y)\,|\Omega\rangle = \dfrac{\text{Numerator}}{\text{Denominator}}$

let us try to evaluate by expanding in powers of λ.

$\text{Numr.} = \langle 0|T\,\varphi_I(x)\,\varphi_I(y)\,e^{-i\int_{-T}^{T} H_I(t')dt'}\,|0\rangle\big|_{\lim T\to\infty(1-i\epsilon)}$

$=$

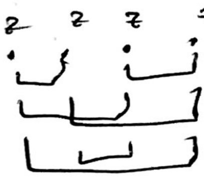
<u>Numerator</u>

$= \langle 0 | T\{ \varphi(x)\varphi(y) - i\varphi(x)\varphi(y) \int_{-T}^{T} dt' H_I(t') dt' + \cdots \} | 0 \rangle$.

$x \overline{\quad\quad} y = D_F(x-y)$

<u>2^{nd} term:</u> $\frac{-i}{4!} \lambda \int d^4z \langle 0 | T \varphi(x)\varphi(y)\varphi^4(z) | 0 \rangle$ use Wick's Thm.

$\langle 0 | \ldots | 0 \rangle = :\varphi(x)\varphi(y)\varphi^4(z): +$ all contractions :



$= \frac{-i\lambda}{4!} \int d^4z \left[3 D_F(x-y) D_F^2(z-z) + 4 D_F(x-z) 3 D_F(y-z) D_F(z-z) \right]$ 3 contractions all give same value

4 of these: x can contr w/ any 4 of the z's. y can contr w/ any of the remaining 3 z's.

$= \frac{1}{8}(-i\lambda) \int d^4z \, D_F^2(z-z) D_F(x-y) + \frac{1}{2}(-i\lambda) \int d^4z \, D_F(x-z) D_F(y-z) D_F(z-z)$

Want to write the expression from the diagram: rules to do so are Feynman rules ~~pt~~ vertex $= -i\lambda \int d^4z$
x & y are fixed but the point z at which the vac fluct happens is arbitrary so integrate over z. Need to put in the symmetry factors assoc to diagram

$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

<u>So summary</u> Numerator Feynman rules

Numerator of 2pt fn $\Big\}$ = Sum of all diagrams with two external points

propagator $D_F(x-y) = x \overline{\quad\quad} y$

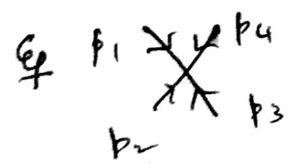
Vertex $-i\lambda \int d^4x$

external line $\overline{\quad\quad}\bullet = 1$

Finally divide by symmetry factor assoc w/ the diagram

<u>Momentum space</u> $D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} D_F(p)$

$D_F(p) = \frac{i}{p^2 - m^2 + i\epsilon} = \xrightarrow{p}$ | vertex $= -i\lambda$; ext line $\overset{\leftarrow}{p} \quad x$ $\; e^{-ip\cdot x}$



Eg $= -i\lambda \int d^4z \, e^{+i(p_1+p_2+p_3+p_4)\cdot z} = -i\lambda (2\pi)^4 \delta^4(\sum_i p_i)$

So we have 4-momentum conservation at each vertex

As before, must divide by symmetry factor

Consider our example

We need to sum over all diagrams. The sum over vacuum bubbles is easy to do

+ $\to$ vacuum bubbles.

Vacuum bubbles: $V_i = \{ \ , \ , \ , \ , \dots \}$. ($V_1$, V_2, V_3, V_4)

a disconn diagram $= \begin{pmatrix}\text{connected}\\ \text{piece}\end{pmatrix} \cdot \prod_i \dfrac{V_i^{n_i}}{n_i!}$ If the V_i th vacuum bubble appears n_i times

Eg connected piece; $n_1 = 2$ & $n_2 = 1$, $n_3 = 1$, $n_4 = 0$.

Numerator $= \sum\limits_{\substack{\text{all possible}\\ \text{connected}\\ \text{pieces}}} \sum\limits_{\substack{\text{all poss}\\ n_i}} (\text{connected piece}) \prod_i \dfrac{1}{n_i!} (V^i)^{n_i}$

each of the terms is of the form $x \quad y$ × vacuum bubbles

Eg $\underset{x \qquad y}{}$ + $\underset{x \qquad y}{}$ + $\underset{x \qquad y}{}$ + $\underset{x \qquad y}{}$ $\longrightarrow$ these are all the connected pieces

each of these conn. pieces is multiplied by a sum of all possible vacuum bubbles

So Numerator $= \sum(\text{Connected}) \sum\limits_{\{\text{all } n_i\}} \prod_i \dfrac{1}{n_i!} (V_i)^{n_i}$

$= \sum(\text{Connected}) \left(\sum\limits_{n_1} \dfrac{1}{n_1!}(V_1)^{n_1}\right)\left(\sum\limits_{n_2} \dfrac{1}{n_2!}(V_2)^{n_2}\right) \dots$

$= \sum(\text{conn}) \prod_i e^{V_i} = \sum(\text{conn})\, e^{\sum_i V_i}$ where V_i are the vacuum diagrams contributions.

So Numerator $= (\text{sum of conn diagrams}) \times e^{(\text{sum of vac. bubb})}$

Eg Num^r $= \left(\underset{x \qquad y}{} + \underset{x \qquad y}{} + \underset{x \qquad y}{} + \underset{x \qquad y}{} + \dots \right) e^{(\ + \ + \ + \dots)}$

Denominator $= \exp(\text{sum of vacuum bubbles}) = \exp(\ + \ + \ + \dots)$

so the exp(sum of vac bubb) cancels betw numerator & denominator

So $\langle \Omega | T \varphi(x) \varphi(y) | \Omega \rangle$ = sum of all connected diagrams with 2 external points x & y (fixed)

$\parallel$

$\underset{x \qquad y}{} = \ + \ + \ + \ + \dots$

Now $\exp(\ + \ + \dots) = |\langle 0 | \Omega \rangle|^2 \exp(-2i E_0 T)$ E_0 = egy of interacting vacuum

So sum of vac. bubbles $= \ + \ + \dots = 2 \log |\langle 0 | \Omega \rangle| - 2i E_0 T$.

$\Rightarrow E_0 + \dfrac{i}{T} \log |\langle 0 | \Omega \rangle| = \dfrac{i}{2T}(\ + \ + \dots)$

So we need the T dependence of the sum of vac diagram

Eg look at $\propto \int d^4z\, d^4w\, D_F(z-z)\, D_F(w-w)\, D_F^2(z-w)$ (upto symm factor)

$$= D_F^2(0) \int d^4z \int d^4w\, D_F^2(z-w)$$

$$\sim D_F^2(0) \int d^4u \underbrace{\int d^4v}\, D_F^2(2u)$$

$u = \frac{z-w}{2}$

$v = \frac{z+w}{2}$

$$\sim D_F^2(0) \cdot (2T) \underbrace{(\text{volume of space})}_{\text{Vol.}} \times \int d^4u\, D_F^2(2u).$$

So Vac bubbles $\propto 2T(\text{Vol of space}) \times (\cdots)$

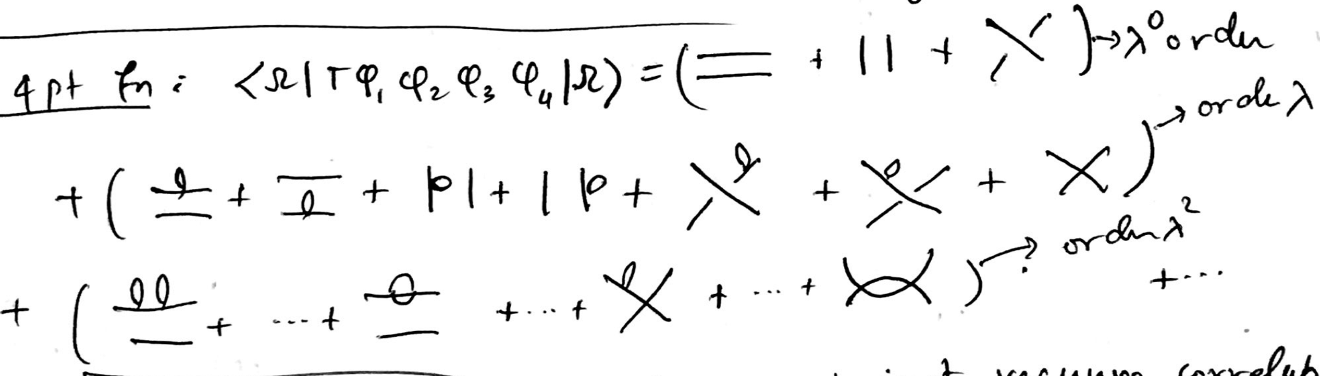
Now $E_0 + \frac{i}{T} \log |\langle 0|\Omega\rangle| = \frac{i}{2T}(8 + \cdots)$

as $T \to \infty$ $\frac{\log|\langle 0|\Omega\rangle|}{T} \to 0$ but contributes

So $\boxed{\frac{E_0}{\text{Vol of sp}} = \frac{i}{2T(\text{Vol})}(8 + \cdots) = \text{energy density in interacting vacuum.}}$

$\boxed{\text{Thus } \langle\Omega|T\varphi(x_1)\cdots\varphi(x_n)|\Omega\rangle = \text{sum of all connected diagrams with } n\text{- external legs.}}$

Eg 4pt fn: $\langle\Omega|T\varphi_1\varphi_2\varphi_3\varphi_4|\Omega\rangle =$



More generally we are interested in not just vacuum correlation fns but correlators between in & out states that have the incoming & outgoing particles → scattering matrix elts

$${}_{\text{out}}\langle \vec{p}_{B_1}\, \vec{p}_{B_2} \cdots \vec{p}_{B_m} | \vec{k}_{A_1}\, \vec{k}_{A_2}\, \vec{k}_{A_2} \cdots \vec{k}_{A_n}\rangle_{\text{in}}$$

$$= {}_0\langle \vec{p}_{B_1} \cdots \vec{p}_{B_m} | S | \vec{k}_{A_1} \cdots \vec{k}_{A_n}\rangle_0 \quad = \text{S matrix elements}$$

$S = I + iT$ Will not give a precise defn of S | simply replace our old formulae with $|0\rangle \to |k \dots k\rangle$

$$\sim \lim_{T\to\infty(1-i\epsilon)} {}_0\langle \vec{p}_{B_1}\, \vec{p}_{B_2} \cdots | T \exp - i\int_{-T}^{T} H_I(t')dt' | \vec{k}_{A_1}\, \vec{k}_{A_2} \cdots \rangle_0.$$

There are subtleties in this which we can see by looking at $2 \to 2$ scattering in ϕ^4 theory

54

look at $\langle \vec{p}_1 \vec{p}_2 | iT | \vec{p}_A \vec{p}_B \rangle = \lim_{T\to\infty(1-i\epsilon)} \left({}_0\langle \vec{p}_1 \vec{p}_2 | T e^{-i\int_{-T}^{T} H_I(t')dt'} | \vec{p}_A \vec{p}_B \rangle \right)$

Connected amputated.

φ^4 theory $2 \to 2$ scattering

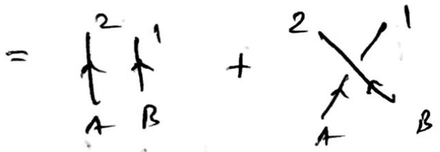
<u>O(1) term:</u> ${}_0\langle \vec{p}_1 \vec{p}_2 | \vec{p}_A \vec{p}_B \rangle = \sqrt{2E_1 2E_2 2E_A 2E_B}\ \langle 0 | a_{p_1} \underbrace{a_{p_2} a^\dagger_{p_A}} a^\dagger_{p_B} | 0 \rangle.$

$$[a(p_2), a^\dagger(p_A)] = (2\pi)^3 \delta^3(p_2 - p_A) + a^\dagger_{p_A} a_{p_2}$$

$= \langle 0 | [a_{p_1}, a^\dagger_{p_B}] | 0 \rangle (2\pi)^3 \delta^3(p_2 - p_A)$

$+ \langle 0 | [a_{p_1}, a^\dagger_{p_A}][a_{p_2}, a^\dagger_{p_B}] | 0 \rangle$

$= (2\pi)^3 \sqrt{2E_1 2E_2 2E_A 2E_B}\ [\delta^3(p_2 - p_A)\delta^3(p_1 - p_B) + \delta^3(p_2 - p_B)\delta^3(p_1 - p_A)]$



This is just free propagation thus does not contribute to T.

<u>At $O(\lambda)$:</u> $\dfrac{-i\lambda}{4!}\ {}_0\langle \vec{p}_1 \vec{p}_2 | T \int d^4z\ \varphi^4(z) | \vec{p}_A \vec{p}_B \rangle_0$

$= \dfrac{-i\lambda}{4!} \int d^4z\ {}_0\langle \vec{p}_1 \vec{p}_2 | :\varphi^4(z) + \text{all contractions}: | \vec{p}_A \vec{p}_B \rangle_0$

$= :\varphi^4(z): + :6\ \overbrace{\varphi(z)\varphi(z)}\ \varphi^2(z) + 3\ \overbrace{\varphi(z)\varphi(z)}\ \overbrace{\varphi(z)\varphi(z)}:$

3rd term $\to$ $= -\dfrac{i\lambda}{4!} 3 \int d^4z\ D_F(z-z)^2\ \underbrace{{}_0\langle p_1 p_2 | p_A p_B \rangle_0} \longrightarrow 8\ \underbrace{(\ \ \ \ )}_{\text{free propagation}}$

No scattering so wont contrib. to T.

2nd term $\to$ $-\dfrac{i\lambda}{4!} \cancel{6}^{\,4} \int d^4z\ D_F(z-z)\ {}_0\langle p_1 p_2 | :\varphi^2(z): | p_A p_B \rangle_0$

a^2 , $(a^\dagger)^2$

$\underline{a^2} + \underline{a^{\dagger 2}} + \cancel{a a^\dagger} + \cancel{a^\dagger a} + 2a^\dagger a$ upon normal ordering

cant contribute non trivially — can contribute

$$-\frac{i\lambda}{2} \int d^4z\ D_F(z-z)\ {}_0\langle \vec{p}_1 \vec{p}_2 | \varphi^-(z) \varphi^+(z) | \vec{p}_A \vec{p}_B \rangle_0 \to \text{to calc this}$$

$\varphi^+(x) | k \rangle = \varphi^+(x) \sqrt{2E_k}\ a^\dagger(k) | 0 \rangle = \sqrt{2E_k} \int \dfrac{d^3q}{(2\pi)^3} \dfrac{1}{\sqrt{2E_q}} [a(q), e^{-iq\cdot x} a^\dagger(k)] | 0 \rangle$

$= \sqrt{2E_k} \int [d^3q] \dfrac{1}{\sqrt{2E_q}} e^{-iq\cdot x} (2\pi)^3 \delta^3(\vec{q} - \vec{k}) | 0 \rangle = e^{-ik\cdot x} | 0 \rangle.$

$= -\dfrac{i\lambda}{2} \int d^4z\ D_F(z-z) [\,{}_0\langle \vec{p}_2 | e^{ip_1 z} + {}_0\langle \vec{p}_1 | e^{ip_2\cdot z}] [e^{-ip_B\cdot z} | \vec{p}_A \rangle_0 + e^{-ip_A\cdot z} | \vec{p}_B \rangle_0]$

$= -\dfrac{i\lambda}{2} \int d^4z\ D_F(z-z) \Big[e^{i(p_1 - p_B)\cdot z} \sqrt{2E_2 2E_A}\ \delta^3(\vec{p}_2 - \vec{p}_A) + e^{i(p_2 - p_B)\cdot z} \sqrt{2E_1 2E_A}\ \delta^3(\vec{p}_1 - \vec{p}_A)$

$+ e^{i(p_1 - p_A)\cdot z} \sqrt{2E_2 2E_B}\ \delta^3(\vec{p}_2 - \vec{p}_B) + e^{i(p_2 - p_A)\cdot z} \sqrt{2E_1 2E_B}\ \delta^3(p_1 - p_B) \Big] (2\pi)^3.$

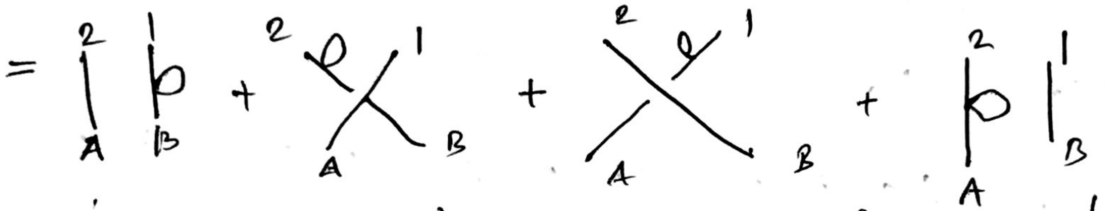
Thus this 2nd term $= \frac{-i\lambda}{2}(2\pi)^3\Big[2E_2\,\delta^3(\vec p_2-\vec p_A)\int_z D_F(z-z)\,e^{i(p_1-p_B)\cdot z}$

$+2E_1\,\delta^3(\vec p_1-\vec p_A)\int_z D_F(z-z)\,e^{i(p_2-p_B)\cdot z}$

$+2E_2\,\delta^3(\vec p_2-\vec p_B)\int_z D_F(z-z)\,e^{i(p_1-p_A)\cdot z}$

$+2E_1\,\delta^3(\vec p_1-\vec p_B)\int_z D_F(z-z)\,e^{i(p_2-p_A)\cdot z}\Big]$

interpret this diagrammatically.



again this describes propagation without interactions betw the particles → still disconnected (part of I, not T)

<u>Finally consider</u> <u>the 1st term</u>

$$\frac{-i}{4!}\int d^4z\ {}_0\langle \vec p_1\,\vec p_2|\underbrace{:\varphi^4(z):}_{6\varphi_-^2\varphi_+^2}|\vec p_A\,\vec p_B\rangle_0 = -i\lambda\int d^4z\ e^{-i(p_A+p_B-p_1-p_2)\cdot z}$$

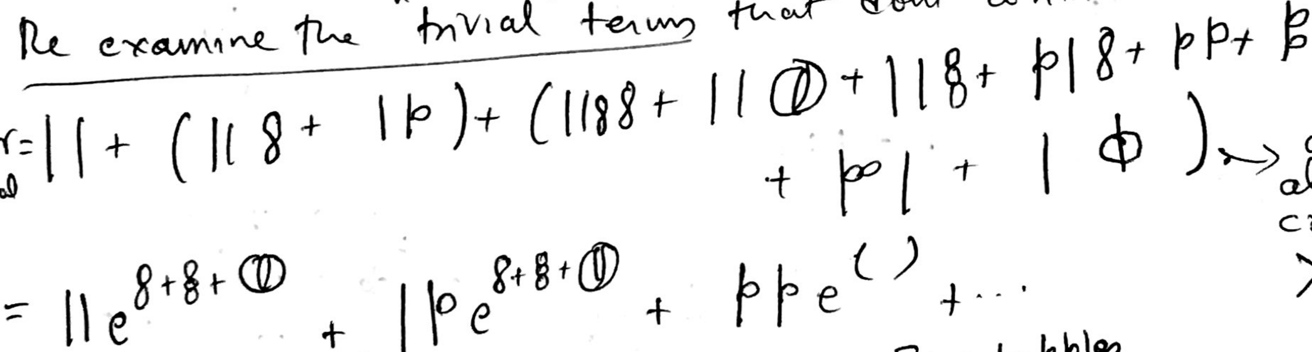
$$= -i\lambda(2\pi)^4\delta^4(p_A+p_B-p_1-p_2)$$

So $iT = (2\pi)^4\delta^4(\text{mom cons})\ i\mathcal{M}$ where $\mathcal{M}=-\lambda$.

This is the only term that contributes to T. at $O(\lambda)$

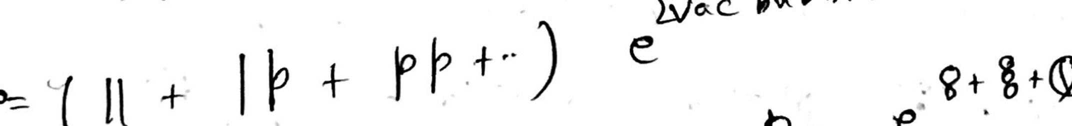
<u>Re examine the "trivial" terms that dont contribute to T.</u>

Numr= trivial



and also the cross versions

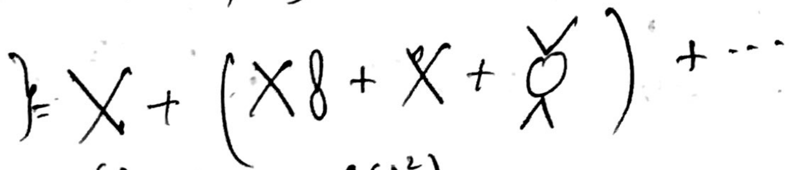
Numerator trivial



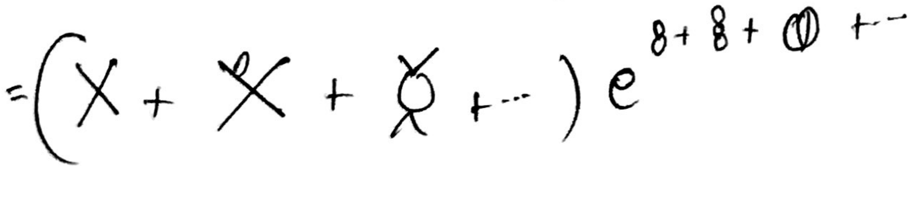
$e^{\Sigma\text{vac bubbles}}$

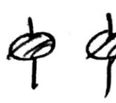
So after dividing by the denominator, the $e^{\dots}$ cancels

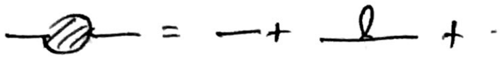
Non trivial terms Numerator

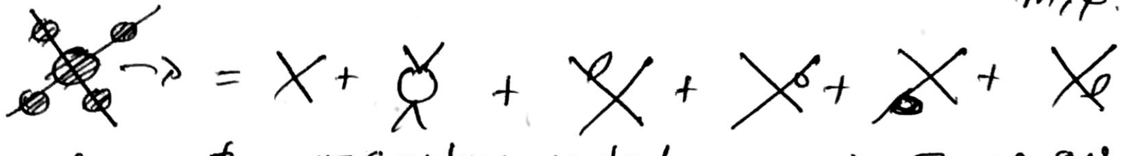


$O(\lambda)$ $\quad$ $O(\lambda^2)$



Renormalization : "Trivial diagrams" =  → disconnected diagrams.

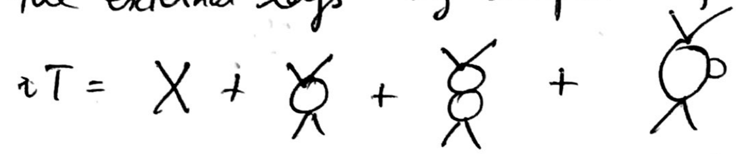
where  this renormalizes the mass & wfn ↓ it is the "full" propagator. m, ϕ.

non-trivial diags = 

This renormalizes the ~~co~~ coupling constant. ↘ These are connected diagrams

What is amputated? Now once you work with the renormalized mass & wfn. 

Since the renormalization of m, ϕ takes care of the decorations on the external legs we can as well drop the external legs by amputating them.

So. 

Tomorrow QED feynman diagrams, SSB, Abelian Higgs mode

Feynman Rules for fermions [Friday 31 Aug 2018]

$$T(\psi(x)\bar\psi(y)) = \begin{cases} \psi(x)\bar\psi(y) & x^0 > y^0 \\ \bar\psi(y)\psi(x) & y^0 > x^0 \end{cases}$$

$$S_F(x-y) = \langle 0|T\psi(x)\bar\psi(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4}\, \frac{i(\not p + m)}{p^2 - m^2 + i\epsilon}\, e^{-ip\cdot(x-y)}$$

Need to pay attention to minus signs in normal & time ordering for Dirac fields. $\left\{ \begin{array}{l} T(\psi_1\psi_2\psi_3\psi_4) = \psi_3\psi_1\psi_4\psi_2 \text{ if } x_3^0 > x_1^0 > x_4^0 > x_2^0 \\ \text{Eg} \end{array}\right.$

"re order the fields in order of time as if they anticommute"

$$:a^\dagger_s(p)\, a_{s'}(q): \;=\; a^\dagger_s(p)\, a_{s'}(q)$$

$$:a_{s'}(q)\, a^\dagger_s(p): \;=\; -a^\dagger_s(p)\, a_{s'}(q).$$

Wick's theorem: $T(\psi(x)\bar\psi(y)) = :\psi(x)\bar\psi(y): + \overbrace{\psi(x)\bar\psi(y)}$

Eg: $:\underbrace{\psi_1\psi_2\bar\psi_3}\bar\psi_4: = -S_F(x_1-x_3) :\psi_2\bar\psi_4:$

QUANTUM ELECTRODYNAMICS

$\mathcal{L} = \bar\psi(i\not D - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ $D_\mu = \partial_\mu + ieA_\mu$ Covariant der.

$$\mathcal{L} = \bar\psi(i\not\partial - e\not A - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \underbrace{\bar\psi(i\not\partial - m)\psi}_{\text{free Dirac}} - \underbrace{\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{free Maxwell}} - \underbrace{e\bar\psi\not A\psi}_{\text{interaction}}.$$

$\mathcal{L}$ has a gauge symmetry $\psi(x) \to e^{i\alpha(x)}\psi(x)$;
$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x)$. check this is an invariance of $\mathcal{L}$.

We say that making the $U(1)_V$ symm of Dirac local (gauging it) forces us to introduce the gauge field A_μ.

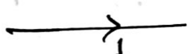
$J^\mu = \bar\psi \gamma^\mu \psi$ noether current of global symm $\Big|\, -e\bar\psi \not{A} \psi = -e A^\mu J_\mu$

$Q = \int d^3x \; e J^0$ = electric charge $\quad Q = e$ for one positron or one proton.

Feynman rules in mom space for QED

vertex comes from $\mathcal{H}_I^{int} = -\mathcal{L}_I^{int}$ and mult by $(-i)$ from Dyson series.


$= -ie\gamma^\mu$

Fermion propagator  $\xrightarrow{p}$ $= \dfrac{i(\not{p}+m)}{p^2 - m^2 + i\epsilon}$

direc of flow of chg negative charge

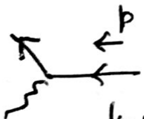
Photon propagator  $\xrightarrow{p}$ $\mu \quad \nu$ $\quad \dfrac{-i\,\eta_{\mu\nu}}{p^2 + i\epsilon}$

External lines: (a) $|\vec p, s\rangle$ particle

$\overbrace{\psi(x)|\vec p, s\rangle} = u_s(\vec p)\, e^{-ip\cdot x}$

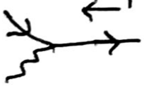
$\psi(x)\sqrt{2E_p}\, a_s^\dagger(\vec p)|0\rangle = \int [d^3q] \sum_{s'} a_{s'}(\vec q)\, u_{s'}(\vec q)\, e^{-iq\cdot x} a_s^\dagger(p)|0\rangle$

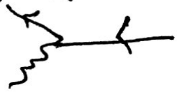
$(2\pi)^3 \delta(p-q)\,\delta_{ss'}$

So  $\xleftarrow{p}$ incoming $|ps\rangle$ particle $= u_s(p)$

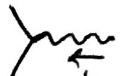
time $\longleftarrow$

(b) Similarly for an incoming antiparticle $\quad \bar\psi\,|\vec p, s\rangle = \bar v(\vec p, s)$

 $\xleftarrow{p}$ $= \bar v(\vec p, s)$

(c) Out-going particle $\quad \underbrace{\langle p s|}\bar\psi = \bar u(\vec p, s) =$  $(\vec p, s)$ p

(d) $\underbrace{\langle p s|}\psi = v(\vec p, s) =$  p $\quad$ outgoing ~~___~~ antiparticle

(e) External photon incoming $\quad \underbrace{A_\mu|\vec p, \lambda\rangle} = \epsilon_\mu(\vec p, \lambda)$ $\quad$ incoming photon  $\vec p, \lambda$ $\quad p$

(f) outgoing photon $\quad \langle \vec p, \lambda| A_\mu = \epsilon_\mu(\vec p, \lambda)$  $\xleftarrow{p}$ $(\vec p, \lambda)$

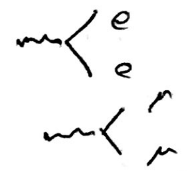
Recall: $\quad S = I + iT \; ; \quad iT = i\mathcal{M}\,(2\pi)^4 \delta^4(\text{mom cons})$

58

Eg: 1) $e^+e^- \to \mu^+\mu^-$

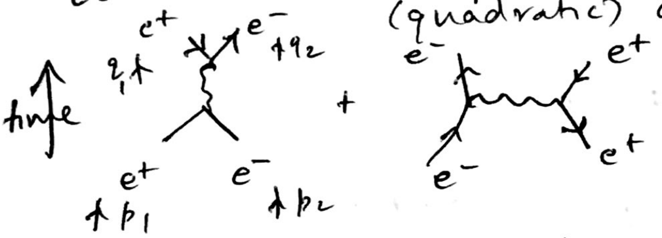
$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}_e (i\not{\partial} - m_e)\Psi_e + \bar{\Psi}_\mu (i\not{\partial} - m_\mu)\Psi_\mu$$

In state: e^-, e^+ out state μ^- & μ^+.



only lowest order diagram is 2nd order in the coupling.

2) Alternatively look at $e^+e^- \to e^+e^-$ elastic scattering. Let time flow upwards. Now there are 2 diagrams at lowest (quadratic) order



Here arrows designate flow of negative charge.

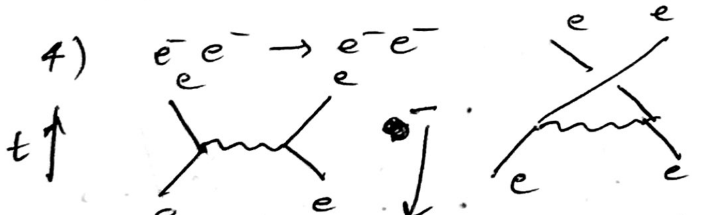
So in a sense $e^+e^- \to \mu^+\mu^-$ is simpler as we have only one diagram to consider at lowest order.

3) $e\gamma \to e\gamma$ elastic scattering. time upwards

Compton scattering

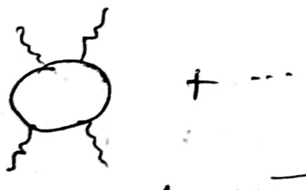
only the sum of the two diagrams is gauge inv.

4) $e^-e^- \to e^-e^-$

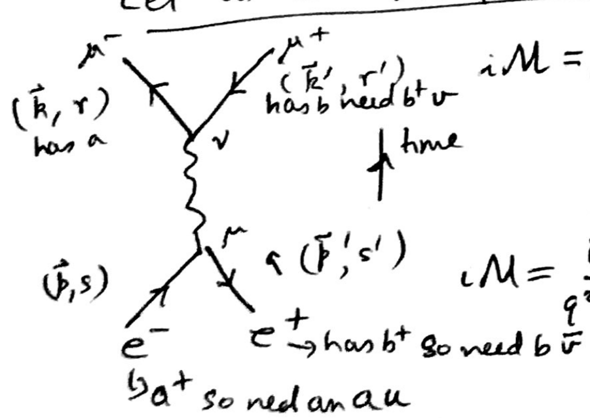


due to exchange of fermion fields

5) $\gamma\gamma \to \gamma\gamma$ No tree level diagram as classical EM does not have photon self interactions



Let us calc $\mathcal{M}$ for $e^+e^- \to \mu^+\mu^-$



$$i\mathcal{M} = \underbrace{\{(-ie)\,\bar{v}_{s'}(\vec{p}')\,\gamma^\mu\, u_s(p)\}}_{\text{incoming vertex}} \underbrace{\frac{-i\eta_{\mu\nu}}{(p+p')^2 + i\epsilon}}_{\text{virtual photon propagator.}} \underbrace{\{(-ie)\,\bar{u}_r(\vec{k})\,\gamma^\nu\, v_{r'}(\vec{k}')\}}_{\text{out going vertex}}$$

$$i\mathcal{M} = \frac{ie^2}{q^2 + i\epsilon}\left(\bar{v}_{s'}(\vec{p}')\gamma^\mu u_s(\vec{p})\right)\left(\bar{u}_r(\vec{k})\gamma_\mu v_{r'}(\vec{k}')\right).$$

$$q = p + p' = k + k'.$$

Spontaneous Symmetry Breaking (SSB)

A field can take a non-zero vacuum expectation value and violate a symmetry of the Lagrangian. Then the theory has undergone SSB.

If the ground state does not possess a symm of the Hamiltonian or Lagrangian then we have SSB.

Eg $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi + \frac{1}{2}\mu^2 \phi^2 - \frac{\lambda}{4!}\phi^4$ with $\mu^2, \lambda > 0$.

has a discrete $\phi \to -\phi$ symmetry

Construct the Hamiltonian $\pi = \frac{\partial \mathcal{L}}{\partial \dot\phi} = \dot\phi$

$$\mathcal{H} = \frac{1}{2}\Big(\underbrace{\dot\phi^2}_{=\pi^2} + (\nabla\phi)^2\Big) - \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4 .$$

ground state config $\phi(x,t) = \phi_0 =$ constant.

$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4$. if $\mu^2 > 0, \lambda > 0$.

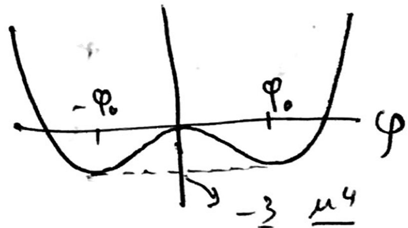
Extrema: $0 = -\mu^2\phi + \frac{\lambda}{3!}\phi^3 = 0 \Rightarrow \phi_0 = 0$ or $\phi_0 = \pm\sqrt{\frac{6}{\lambda}}\,\mu$

Check whether maxima or minima: $\frac{\partial^2 V}{\partial\phi^2} = -\mu^2 + \frac{\lambda}{2!}\phi^2$

$\varphi_0 = 0 \Rightarrow -\mu^2 = V'' \to$ maximum

$\varphi_0 = \pm\mu\sqrt{\frac{6}{\lambda}} \Rightarrow -\mu^2 + \frac{\lambda}{2}\mu^2\frac{6}{\lambda} = 2\mu^2 \to$ minimum.

$$V\Big(\underbrace{\pm\mu\sqrt{\tfrac{6}{\lambda}}}_{\varphi_0}\Big) = -\frac{1}{2}\mu^2\varphi_0^2 + \frac{\lambda}{4!}\varphi_0^4 = -\frac{3}{2}\frac{\mu^4}{\lambda}$$



$\boxed{\varphi_0 = \mu\sqrt{\frac{6}{\lambda}}}$

other remarks

Explicit symm breaking is if you add a φ^3 term for instance that explicitly brk $\varphi \to -\varphi$.

Note a vector field cannot have a non zero vev

$\langle 0|A_\mu|0\rangle \neq 0$ would violate Lorentz inv

"Dynamical symm breaking" is a name used for when the symm is unbroken to all orders in perturbation theory but is broken but non-pert (instanton) effects

Now to quantize we do perturbation theory around $\varphi = \varphi_0$.

Let $\varphi(x) = \varphi_0 + \sigma(x)$ and expand $\mathcal{L}$.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{\mu^2}{2}(\varphi_0+\sigma)^2 - \frac{\lambda}{4!}(\varphi_0+\sigma)^4$$

$$= \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{\mu^2}{2}(\varphi_0^2 + \sigma^2 + 2\varphi_0\sigma) - \frac{\lambda}{4!}(\varphi_0^4 + 4\varphi_0^3\sigma + 6\varphi_0^2\sigma^2 + 4\varphi_0\sigma^3 + \sigma^4)$$

Let us drop additive constants

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 + \underbrace{\sigma^2\left(\frac{\mu^2}{2} - \frac{\lambda}{4}\varphi_0^2\right)}_{\substack{\frac{\mu^2}{2} - \frac{\lambda}{4}\mu^2\frac{6}{\lambda} \\ \underbrace{\qquad}_{=\,-\mu^2}}} + \underbrace{\sigma\left(\mu^2\varphi_0 - \frac{\lambda}{6}\varphi_0^3\right)}_{\varphi_0\left(\mu^2 - \frac{\lambda\varphi_0}{6}\right) \to 0} - \frac{\lambda}{6}\varphi_0\sigma^3 - \frac{\lambda}{4!}\sigma^4.$$

~~So~~

expanding around a stable vacuum { notice this sign change ← $-\mu^2$ $\boxed{m = \sqrt{2}\mu}$

→ not surprising as we are expanding around an extremum (minimum $V'(\varphi_0) = 0$)

So
$$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{2}m^2\sigma^2 - \frac{\lambda}{6}\sqrt{\frac{6}{\lambda}}\,\mu\,\sigma^3 - \frac{\lambda}{4!}\sigma^4.$$

$$\text{———} = \frac{i}{p^2 - m^2 + i\epsilon}$$ Y cubic X quartic.

So far we looked at breaking of global discrete symmetry.
Let us go to a global continuous symmetry - breaking.

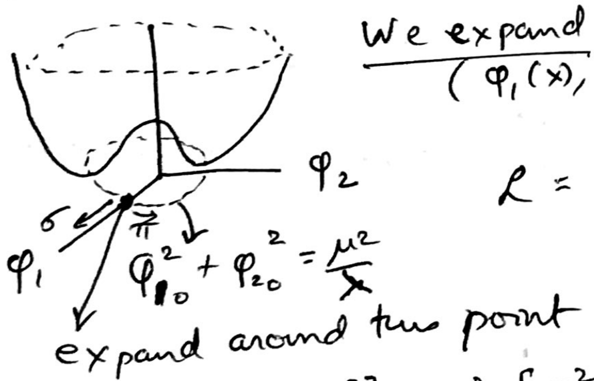
Complex scalar field with $\mu^2, \lambda > 0$.

$$\mathcal{L} = \partial_\mu\varphi^*\partial^\mu\varphi + \mu^2\varphi^*\varphi - \lambda(\varphi^*\varphi)^2$$

$\varphi \to e^{i\alpha}\varphi$ is a global symm $\quad \varphi = \dfrac{\varphi_1 + i\varphi_2}{\sqrt{2}}$

$$V(\varphi,\varphi^*) = -\mu^2\varphi^*\varphi + \lambda(\varphi^*\varphi)^2 \Rightarrow \frac{\partial V}{\partial\varphi^*} = -\mu^2\varphi + 2\lambda(\varphi^*\varphi)\varphi = 0$$

So extrema are at $|\varphi_0|^2 = \dfrac{\mu^2}{2\lambda} = \dfrac{\varphi_{10}^2 + \varphi_{20}^2}{2}$. Can plot $V(\varphi,\varphi^*)$



We expand $(\varphi_1(x), \varphi_2(x)) = \left(\frac{\mu}{\sqrt{\lambda}} + \sigma(x),\ \pi(x)\right).$

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu\sigma)^2 + (\partial_\mu\pi)^2\right] + \frac{\mu^2}{2}\left[\pi^2 + \left(\frac{\mu}{\sqrt{\lambda}} + \sigma\right)^2\right] - \frac{\lambda}{4}\left[\pi^2 + \left(\frac{\mu}{\sqrt{\lambda}} + \sigma\right)^2\right]^2$$

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu\sigma)^2 + (\partial_\mu\pi)^2\right] + \frac{\mu^2}{2}\left[\pi^2 + \sigma^2 + 2\sigma\frac{\mu}{\sqrt{\lambda}} + \frac{\mu^2}{\lambda}\right] - \frac{\lambda}{4}\left[\pi^4 + \sigma^4 + 4\sigma^2\frac{\mu^2}{\lambda} + 2\sigma^2\pi^2 + 4\pi^2\sigma\frac{\mu}{\sqrt{\lambda}} + 2\pi^2\frac{\mu^2}{\lambda} + 4\sigma^3\frac{\mu}{\sqrt{\lambda}} + 2\sigma^2\frac{\mu^2}{\lambda} + \frac{4\mu^3\sigma}{\lambda\sqrt{\lambda}}\right]$$

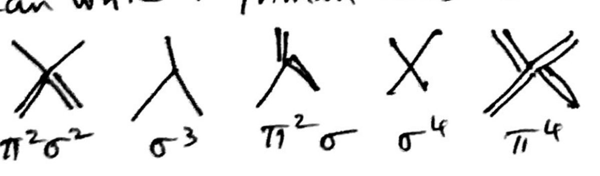
Now drop all additive constants. Linear terms in σ cancel.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\pi)^2 - \frac{1}{2}(2\mu^2)\sigma^2 - \frac{\lambda}{4}(\pi^4 + \sigma^4) - \frac{\lambda}{2}\pi^2\sigma^2 - \mu\sqrt{\lambda}\,\sigma^3 - \mu\sqrt{\lambda}\,\pi^2\sigma$$

The dynamical fields are σ and π. field σ has got a mass term

$m_\sigma = \sqrt{2}\mu \qquad m_\pi = 0 \quad \to$ no mass term for π field.

Can write Feynman rules $\pi \Longrightarrow \dfrac{i}{p^2 + i\epsilon} \qquad \sigma \longrightarrow \dfrac{i}{p^2 - m^2 + i\epsilon} \qquad m = \sqrt{2}\mu.$



vertices. π is called Nambu Goldstone boson
Generally every spontaneously broken global continuous symm generator gives a NG boson.

Eg if you have 3 NG bosons $\quad \sigma^2 + \vec{\pi}^2 = (\text{const})^2$ defines the vacua.
(N-1) NG bosons comes from O(N) non linear sigma model where O(N) symm is broken to O(N-1). The pions are massless Nambu Goldstone bosons

Gauge the global U(1) symmetry → local U(1)

$$\mathcal{L} = (D_\mu\phi)^* D^\mu \phi + \mu^2 \varphi^*\varphi - \lambda(\varphi^*\varphi)^2 - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \quad \Big| \quad D_\mu = \partial_\mu + ieA_\mu$$

gauge symm: $\varphi(x) \to e^{i\alpha(x)}\varphi(x)$, $\quad A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha(x)$.

$|\varphi_0| = \frac{\mu}{\sqrt{\lambda}}$, $A_\mu^0 = 0$. $\quad$ $\left.\begin{aligned}\varphi_1'(x) &= \varphi_1 - \mu/\sqrt{\lambda} \\ \varphi_2'(x) &= \varphi_2(x)\end{aligned}\right\}$ introduce fluctuating fields

we find

$m_{\varphi_1'} = \sqrt{2}\mu$ $\quad$ & $\quad m_{\varphi_2'} = 0 \to$ massless. $\quad$ Look at gauge field interactions

$$\mathcal{L} = (\partial_\mu\varphi + ieA_\mu\varphi)(\partial^\mu\varphi^* - ieA^\mu\varphi^*) + \cdots$$

$$= \frac{1}{2}(\partial_\mu\varphi_1)^2 + \frac{1}{2}(\partial_\mu\varphi_2)^2 - ieA^\mu(\varphi^*\partial_\mu\varphi - \varphi\partial_\mu\varphi^*) + \frac{e^2}{2}A^\mu A_\mu(\varphi_1^2 + \varphi_2^2) + \cdots$$

$$= \frac{1}{2}(\partial_\mu\varphi_1')^2 + \frac{1}{2}(\partial_\mu\varphi_2')^2 + eA^\mu\left[-\varphi_2'\partial_\mu\varphi_1' + \left(\frac{\mu}{\sqrt{\lambda}} + \varphi_1'\right)\partial_\mu\varphi_2'\right]$$

$$+ \frac{e^2}{2}A^\mu A_\mu\left[(\varphi_1'^2 + \varphi_2'^2) + \frac{2\mu}{\sqrt{\lambda}}\varphi_1'\right] + \cdots$$

focus on some photon mass terms & another one linear in A_μ & φ

$$\underbrace{\frac{e^2\mu^2}{2\lambda}A^\mu A_\mu}_{\text{photon mass term}} + \frac{e\mu}{\sqrt{\lambda}}A^\mu\partial_\mu\varphi_2'$$

$m_A = \frac{e\mu}{\sqrt{\lambda}}$

derivative coupling

This is a model for superconductivity $\quad e^-e^-$ condensate
Charged condensate breaks the U(1) gauge symmetry
leads to Meissner effect.

$\boxed{\text{Higgs-Englert-Brout mech}}$

Can remove φ_2' from spectrum and get a modified propagator
φ_2' gives a longitudinal mode to the photon
$\varphi_1' \to$ Higgs boson; $\quad (A_\mu, \varphi_1') \to$ together massive photon.

Can make this explicit in Unitary gauge.

$\varphi(x) = \frac{1}{\sqrt{2}}\left(\frac{\mu}{\sqrt{\lambda}} + h(x)\right)\exp\left[i\frac{\xi(x)}{\mu/\sqrt{\lambda}}\right].$ $|\varphi(x)|_{vac} = \frac{\mu^2}{2\lambda}.$

$\frac{1}{\sqrt{2}}\left(\frac{\mu}{\sqrt{\lambda}} + h\right)\left(1 - \frac{i\xi}{\mu/\sqrt{\lambda}}\right).$ $h(x)$ & $\xi(x)$ are the fluctuating fields.

Now make a gauge transformation by param $\alpha = -\xi(x)/(\mu/\sqrt{\lambda})$.

$\varphi'(x) = \varphi(x)\exp[-i\xi(x)/(\mu/\sqrt{\lambda})]$
$A_\mu'(x) = A_\mu(x) + \frac{1}{e}\partial_\mu \xi(x)/(\mu/\sqrt{\lambda})$ | Now write the $\mathcal{L}$ in terms of primed fields.

2 d.o.f of A & 1 d.o.f of ξ have combined to give a massive photon.

$(A_\mu', h) \to$ new fluctuating fields massive photon & massive higgs fld.

Tomorrow: Non-abelian gauge theory.

Saturday 1 Sep: Classical gauge theory.

Reconsider QED: $\mathcal{L} = \bar\psi(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$; $D_\mu = \partial_\mu + ieA_\mu$.

$\psi(x) \to e^{i\alpha(x)}\psi(x)$, $\bar\psi(x) \to e^{-i\alpha(x)}\bar\psi(x)$, $A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha(x)$.

Geometry of gauge invariance. Consider Dirac fermions that transform as $\psi(x) \to e^{i\alpha(x)}\psi(x)$, $\bar\psi(x) \to e^{-i\alpha(x)}\bar\psi(x)$.

Then $m\bar\psi(x)\psi(x)$ is invariant. What about the derivative?

Consider directional deriv $n^\mu\partial_\mu\psi(x) = \lim_{\epsilon\to 0}\frac{\psi(x^\mu + \epsilon n^\mu) - \psi(x^\mu)}{\epsilon}$

n^μ $\psi(x)$ & $\psi(x+\epsilon n)$ transform via different phases

So $n^\mu\partial_\mu\psi$ does not transform in a definite way.

So we introduce a compensator $U(x,y)$ and postulate that it transforms according to

$U(x,y) \to e^{i\alpha(x)}U(x,y)e^{-i\alpha(y)}$ with $U(x,x) = 1$.

Then $U(x,y)\psi(y) \to e^{i\alpha(x)}U(x,y)e^{-i\alpha(y)}e^{i\alpha(y)}\psi(y) = e^{i\alpha(x)}U(x,y)\psi(y)$

So $U\psi$ transforms in the same way as $\psi(x)$ for any y. Use this to define a covariant derivative

$n^\mu D_\mu\psi(x) = \lim_{\epsilon\to 0}\frac{\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)}{\epsilon}$ here both $\psi(x+\epsilon n)$ & $U\cdot\psi$ transform in the same way.

for small ϵ, $U(x+\epsilon n, x) = 1 - i\epsilon e n^\mu A_\mu(x)$. define $\boxed{-ieA_\mu \equiv i\partial_\mu U(x,x)}$

$1 + i\epsilon n^\mu\partial_\mu U(x,x) + O(\epsilon^2)$ $\boxed{A_\mu = -\frac{1}{e}\partial_\mu U}$

Thus:

$n^\mu D_\mu\psi(x) = \lim_{\epsilon\to 0}\frac{\psi(x) + \epsilon n^\mu\partial_\mu\psi(x) - \psi(x) + ie\epsilon n^\mu A_\mu\psi(x)}{\epsilon}$

$= n^\mu(\partial_\mu + ieA_\mu)\psi = n^\mu D_\mu\psi$. where D_μ was the covariant der intro earlier.

Let us see how $U(x+\epsilon n, x)$ transforms & how A_μ transf.

$$U(x+\epsilon n, x) \to e^{i\alpha(x+\epsilon n)}\, U(x+\epsilon n, x)\, e^{-i\alpha(x+\cancel{\epsilon})}$$

as $U(x,y) \to e^{i\alpha(x)}\, U(x,y)\, u^{-\alpha(y)}$

So $1 - ie\epsilon n^\mu A_\mu(x) \to (1 - ie\epsilon n^\mu A_\mu(x))\,(1 \underbrace{- i\alpha(x) + i\alpha(x+\epsilon n)}_{+ i\epsilon n^\mu \partial_\mu \alpha(x)})$

keeping terms only to $O(\epsilon)$,

$$1 - ie\epsilon n^\mu A_\mu(x) \to 1 + i\epsilon n^\mu \partial_\mu \alpha(x) - ie\epsilon n^\mu A_\mu(x)$$

So $A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$ so we recover the known transformation property of $A_\mu(x)$

Thus $\psi(x) \to e^{i\alpha(x)}\psi(x)$ and $D_\mu \psi(x) \to e^{i\alpha(x)} D_\mu \psi(x)$ transforms in the same way. thats why it is called covariant der.

It follows that $D_\nu D_\mu \psi(x) \to e^{i\alpha(x)} D_\nu D_\mu \psi(x)$

Moreover, $[D_\mu, D_\nu]\psi(x) = [\partial_\mu + ieA_\mu, \partial_\nu + ieA_\nu]\psi(x)$.

$$= ie(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi(x)$$

$$= ieF_{\mu\nu}(x)\psi(x) \to ie F'_{\mu\nu}(x)\, e^{i\alpha(x)}\psi(x)$$

Under a gauge transformation, $F_{\mu\nu}$ is invariant since $[D,D]\psi$ and ψ both transf by a prefactor phase $e^{i\alpha(x)}$.

i.e. $[D_\mu, D_\nu]\psi(x) \to e^{i\alpha(x)}[D_\mu, D_\nu]\psi(x)$ & $\psi(x) \to e^{i\alpha(x)}\psi(x)$

together implies $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is gauge invariant.

Now we can solve $A_\mu = -\frac{1}{e}\partial_\mu u$ to get

$$U(x,y) = e^{-ie\int_{y^\mu}^{x^\mu} dz^\nu A_\nu(z)} \longleftrightarrow e^{-ie\int_{y^\mu}^{x^\mu} dz^\nu (A_\nu(z) - \frac{1}{e}\partial_\nu \alpha(z))}$$

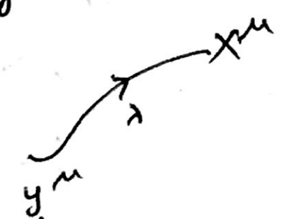
$$= U(x,y)\, e^{i\int_{y^\mu}^{x^\mu} \partial_\nu \alpha(z)} = U(x,y)\, e^{i\int_y^x dz^\nu \partial_\nu \alpha(z)}$$

$$= U(x,y)\, e^{i[\alpha(x) - \alpha(y)]} = e^{i\alpha(x)}\, U(x,y)\, e^{-i\alpha(y)}.$$

$U(x,y)$ is called the Wilson "line" from y to x

" $\exp -ie\int_y^x \frac{dz^\nu}{d\lambda} A_\nu(z(\lambda))\, d\lambda$

↳ line integral



$U(x,x) =$ is a Wilson line around a closed curve

↳ Wilson loop → it is gauge invariant. It is not simply 1. it depends on the curve from x back to x

$U(x,y)$ depends on the curve from y to x. not just on end pts.
QED is an abelian gauge theory. $U(1)$ phase changes.

Non abelian gauge theory / Yang-Mills theory + matter

Consider Dirac fermions ψ transforming under an irreducible representation r of a semi-simple group G with hermitian generators T_a in the representation r.
[Semisimple means no abelian normal (invariant) subgroup].

Now consider Global G transformations.

$$\psi_i(x) \to \psi'_i(x) = \left(e^{i\vec{\theta}\cdot\vec{T}}\right)_{ij}\psi_j(x)$$ if $SU(2)$ spin $\frac{1}{2}$ rep then $\vec{T} = \vec{\sigma}$.

$\theta^a \to$ constants, parametrize the transformation.
$T^a \to$ hermitian generators $a = 1, \ldots, \dim G$.
$i, j = 1, \ldots, \dim(\text{repn } r)$.

$[T^a, T^b] = i f^{abc} T^c$ f^{abc} can be taken totally antisymm
$\hookrightarrow$ structure constants.

start with

$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$. $\psi(x) \to e^{i\vec{\theta}\cdot\vec{T}}\psi(x)$ & $\bar{\psi}(x) \to \bar{\psi}(x)e^{-i\vec{\theta}\cdot\vec{T}}$.

global symm $\Rightarrow$ cons. current. $\delta S = \int d^4x\, \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial \partial_\mu\phi}\delta\phi\right]$.
$$= \int d^4x\, \partial_\mu\left[\bar{\psi}\gamma^\mu T^a\psi\right]\theta^a$$

So $\vec{J}^\mu = \bar{\psi}\gamma^\mu\vec{T}\psi$. So we have $\dim(G)$ number of conserved currents. $J^{a\mu}$.

<u>Gauge the symmetry G</u> $\theta^a \to \theta^a(x^\mu)$

$$\psi(x) \to \psi'(x) = e^{i\vec{\theta}(x)\cdot\vec{T}}\psi(x) = U(\theta)\psi(x)$$
$\hookrightarrow$ unitary $\boxed{U^\dagger = U^{-1}}$

Now look at transformation property of $\bar{\psi}_{\alpha i}(x)\gamma^\mu_{\alpha\beta}\partial_\mu\psi_{\beta i}(x) \to \bar{\psi}U^\dagger\gamma^\mu\partial_\mu(U\psi)$
$$= \bar{\psi}\gamma^\mu\partial_\mu\psi + \bar{\psi}\gamma^\mu(U^{-1}\partial_\mu U)\psi$$

$\alpha \to$ Dirac spinor index
$i, j \to$ label repn space

Now as in QED, introduce a covariant derivative that transf in the same way as ψ. $D_\mu\psi(x) \equiv (\partial_\mu - ig\vec{T}\cdot\vec{A}_\mu)\psi$.

$$(D_\mu\psi)_i = (\partial_\mu\psi)_i - ig(T^a)_{ij}A^a_\mu\psi_j(x)$$ $a = 1, \ldots, \dim(G)$.
$\swarrow$
$i = 1, \ldots, \dim(r)$

$\psi(x) \to \dim(r) \times 1$ column vector

<u>Now</u>: $D_\mu\psi \to (D_\mu\psi)' = U(\theta)(D_\mu\psi) \to$ we want this. Question how should A_μ transform to get this?

So $(\partial_\mu - ig\vec{T}\cdot\vec{A}'_\mu)(U\psi) = U(\partial_\mu - ig\vec{T}\cdot\vec{A}_\mu)\psi \Rightarrow$

$$(\partial_\mu U)\psi + U\partial_\mu\psi - ig\vec{T}\cdot\vec{A}'_\mu U\psi = U\partial_\mu\psi - igU\vec{T}\cdot\vec{A}_\mu\psi.$$

$$\Rightarrow \vec{T}\cdot\vec{A}'_\mu = -\frac{i}{g}(\partial_\mu U)U^{-1} + U\vec{T}\cdot\vec{A}_\mu U^{-1}$$

Thus: $\vec{T}\cdot\vec{A}'_\mu(x) = U\,\vec{T}\cdot\vec{A}_\mu(x)\,U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}$

This is how A_μ transforms under finite gauge transformations

Now consider infinitesimal transformations $U \simeq 1 + i\vec{\theta}\cdot\vec{T}$

$$\vec{T}\cdot\vec{A}'_\mu = (1+i\vec{\theta}\cdot\vec{T})\,\vec{T}\cdot\vec{A}_\mu\,(1-i\vec{\theta}\cdot\vec{T}) - \frac{i}{g}\,i(\vec{T}\cdot\partial_\mu\vec{\theta})(1-i\vec{\theta}\cdot\vec{T})$$

$$= \vec{T}\cdot\vec{A}_\mu - i[\vec{A}_\mu\cdot\vec{T}, \vec{\theta}\cdot\vec{T}] + \frac{1}{g}\vec{T}\cdot(\partial_\mu\vec{\theta})$$

or in component form.

$$T^a A'^a_\mu = T^a A^a_\mu + \frac{1}{g}T^a(\partial_\mu\theta^a) - i A^b_\mu \theta^c \underbrace{[T^b, T^c]}_{i f^{abc} T^a}$$

$$\Rightarrow \boxed{A'^a_\mu = A^a_\mu + \frac{1}{g}\partial_\mu\theta^a + f^{abc}A^b_\mu\theta^c} \quad (*)$$

↳ absent in QED or any abelian gauge theory.

Now look at a field Φ that transforms in the adjoint rep of G:

$$\Phi \to \Phi' = e^{i\vec{\theta}\cdot\vec{T}}\Phi$$

$\delta\Phi_i = i\,\vec{\theta}\,\vec{T}_{ij}\,\Phi_j$. where $(T^a)_{bc} = i f_{abc}$.

$\delta\Phi_a = i\,\theta^b \underbrace{(T^b)_{ac}}_{i f_{abc}} \Phi_c$ so $\delta\Phi_a = -f_{abc}\theta_b\Phi_c$

$= f_{abc}\theta_c\Phi_b$

So comparing with $(*)$ above, we say that A^a_μ transform in the adjoint repn (aside from the $\frac{1}{g}\partial_\mu\theta^a$ term)

So we say that the gluons of QCD are charged under SU(3) color. Now lets see how $[D_\mu, D_\nu]$ transforms, first calc it:

$$[D_\mu, D_\nu] = -ig\,\vec{T}\cdot\vec{F}_{\mu\nu} = [\partial_\mu - ig\vec{T}\cdot\vec{A}_\mu, \partial_\nu - ig\vec{T}\cdot\vec{A}_\nu]$$

$$= -ig\,\vec{T}\cdot\partial_\mu\vec{A}_\nu + ig\,\vec{T}\cdot\partial_\nu\vec{A}_\mu - g^2[\vec{T}\cdot\vec{A}_\mu, \vec{T}\cdot\vec{A}_\nu]$$

$$\Rightarrow -ig\,\vec{T}\cdot\vec{F}_{\mu\nu} = -ig\,\vec{T}\cdot(\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu) - g^2[\vec{T}\cdot\vec{A}_\mu, \vec{T}\cdot\vec{A}_\nu]$$

$$-ig\,\vec{T}\cdot\vec{F}_{\mu\nu} = -ig\,T^a(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) - g^2 A^b_\mu A^c_\nu \underbrace{[T^b, T^c]}_{i f^{abc} T^a}$$

So $$\boxed{F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu}$$

let us check how $[D_\mu, D_\nu]\psi$ transforms

$\psi \to U\psi$, $\quad D_\mu\psi \to U D_\mu\psi \quad D_\nu D_\mu\psi \to U D_\nu D_\mu\psi$.

then $[D_\mu, D_\nu]\psi \to U[D_\mu, D_\nu]\psi = ([D_\mu, D_\nu]\psi)'$

Thus $\vec{T}\cdot\vec{F}'_{\mu\nu}\, U = U\, \vec{T}\cdot F_{\mu\nu} \Rightarrow \vec{T}\cdot\vec{F}'_{\mu\nu} = U(T\cdot F_{\mu\nu})\, U^{-1}$

Thus $F^a_{\mu\nu}T^a$ is not gauge invariant, unlike the field strength of QED. Next look at how it transforms under infinitesimal gauge transformation:

$$\vec{T}\cdot\vec{F}'_{\mu\nu} = (1 + i\vec{\theta}\cdot\vec{T})\, \vec{T}\cdot\vec{F}_{\mu\nu}\, (1 - i\vec{\theta}\cdot\vec{T})$$

$$= \vec{T}\cdot\vec{F}_{\mu\nu} - i[\vec{T}\cdot\vec{F}_{\mu\nu}, \vec{\theta}\cdot\vec{T}].$$

So $T^a F^{a\prime}_{\mu\nu} = T^a F^a_{\mu\nu} - i F^b_{\mu\nu}\theta^c \underbrace{[T^b, T^c]}_{\to\, if^{abc}T^a}$.

$\Rightarrow$ under infinitesimal gauge transformations

$$\boxed{F'^a_{\mu\nu} = F^a_{\mu\nu} + \theta^b F^c_{\mu\nu} f^{abc}} \Rightarrow \boxed{\delta F^a_{\mu\nu} = f^{abc}\theta^b F^c_{\mu\nu}}$$

How to construct a gauge invariant action? Take trace

$\vec{T}\cdot\vec{F}'_{\mu\nu} = U(T\cdot F_{\mu\nu})U^{-1}$ ~ how $T\cdot F_{\mu\nu}$ transforms

Thus $tr(\vec{T}\cdot\vec{F}_{\mu\nu}\, \vec{T}\cdot F^{\mu\nu}) \to tr(\cancel{U}\, T\cdot F_{\mu\nu}\cancel{U^{-1}}\cancel{U}\, T\cdot F^{\mu\nu}\cancel{U^{-1}})$

$= tr(T\cdot F_{\mu\nu}\, T\cdot F^{\mu\nu})$ is gauge inv.

In components. $tr(\vec{T}\cdot\vec{F}_{\mu\nu}\, \vec{T}\cdot F^{\mu\nu}) = \underbrace{tr(T^aT^b)}_{C(r)\delta^{ab}}(F^a_{\mu\nu}F^{\mu\nu b})$

↳ depends on the representation

if r = fund rep, $C(r) = \frac{1}{2}$ can be

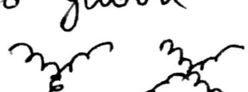
so $\boxed{-\frac{1}{2}\, tr(\vec{T}\cdot F_{\mu\nu}\, \vec{T}\cdot F^{\mu\nu}) = -\frac{1}{4} F^a_{\mu\nu}F^{\mu\nu a}}$.

So a gauge & Lorentz invariant action is

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\Psi}(i\cancel{D} - m)\Psi + (D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi)$$

$$D_\mu\phi_i = \partial_\mu\phi_i - ig(T^a)_{ij}\phi_j$$

This Lagrangian describes gluon propagator & also gluon self interactions (unlike for the photon)



<u>Note</u>: to define the Wilson line, need to use path ordered exponential

$$P[e^{ig\int d\lambda \frac{dz^\nu}{d\lambda} A^a_\nu(z(\lambda))T^a}] = \text{Wilson line.}$$

↳ path ordering

Remark on Noether's thm & Belinfante construction

$\delta S = \int d^4x\, \partial_\mu[\delta x^\mu \mathcal{L} + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi]$ Translations $\phi'(x') = \phi(x)$

$\delta\phi(x) = -a^\mu\partial_\mu\phi(x)$ $\quad \psi'_\alpha(x') = \psi_\alpha(x) \Rightarrow \delta\psi_\alpha = -a^\mu\partial_\mu\psi_\alpha(x)$

$A'_\mu(x') = A_\mu(x) \Rightarrow \delta A_\mu(x) = -a^\nu\partial_\nu A_\mu(x)$

$T_{\mu\nu} = -\eta_{\mu\nu}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\nu \phi$. To calc the spin of Dirac & Maxwell we used:

for spin $\frac{1}{2}$ used $x_\alpha T_{\mu\beta} - x_\beta T_{\mu\alpha}$ → Noether stress tensor (non symm)

But for spin 1 we used the Belinfante tensor ↳ symmetric

Why is this ok?

Under Lorentz transformations $\phi'(x^\mu + \omega^\mu{}_\nu x^\nu) = \phi(x)$

$$\delta\phi(x) = \omega^{\mu\nu} x_\nu \partial_\mu \phi(x)$$

$$V'^\mu(x^\alpha + \omega^\alpha{}_\beta x^\beta) = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu(x)$$

$$\Rightarrow \delta V^\mu(x) = \omega^{\alpha\beta} x_\beta \partial_\alpha V^\mu(x) + \omega^\mu{}_\nu V^\nu(x) =$$

$$= -\frac{i}{2}\omega^{\alpha\beta}(J_{\alpha\beta})^\mu{}_\nu V^\nu(x)$$

$\vec{J} = \vec{L} + \vec{S}$ $\quad (S_{\alpha\beta})^\mu{}_\nu = i(\delta^\mu{}_\alpha \eta_{\nu\beta} - \delta^\mu{}_\beta \eta_{\nu\alpha})$; $\quad S_{\mu\nu} = \frac{\sigma_{\mu\nu}}{2}$

$\psi'(x') = e^{-\frac{i}{4}\omega^{\alpha\beta}\sigma_{\alpha\beta}}\psi(x)$ where $(\sigma_{\mu\nu})^\alpha{}_\beta = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$

$$\partial_\mu\left[\omega^\mu{}_\nu x^\nu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\left\{\omega^{\lambda\rho} x_\lambda \partial_\rho \phi - \frac{i}{2}\omega^{\lambda\rho} S_{\lambda\rho}\phi\right\}\right]$$

$$-J^\mu{}_{\lambda\rho} = x_\lambda T^\mu{}_\rho - x_\rho T^\mu{}_\lambda + S^\mu{}_{\lambda\rho} - i\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} S_{\lambda\rho}\phi$$

$$\partial_\mu S^\mu{}_{\lambda\rho} = T_{\lambda\rho} - T_{\rho\lambda}$$

Belinfante-Rosenfeld stress tensor

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda(S^{\mu\nu\lambda} + S^{\nu\mu\lambda} - S^{\lambda\nu\mu}).$$

$$-J^\mu{}_{\nu\rho} = x_\lambda \tilde{T}^\mu{}_\rho - x_\rho \tilde{T}^\mu{}_\lambda + \frac{1}{2}\partial_\nu\left[x_\rho(S^\mu{}_\lambda{}^\nu + S_\lambda{}^{\mu\nu} - S^\nu{}_\lambda{}^\mu)\right]$$

$$- \frac{1}{2}\partial_\nu\left[x_\lambda(S^\mu{}_\rho{}^\nu + S_\rho{}^{\mu\nu} - S^\nu{}_\rho{}^\mu)\right].$$

These 2 last terms do not affect the charge $\int J^0{}_{\nu\rho}\, d^3x$. (total der)

This justifies our earlier calculation

$$-J^\mu{}_{\lambda\rho} = x_\lambda \tilde{T}^\mu{}_\rho - x_\rho \tilde{T}^\mu{}_\lambda$$

Maxwell: $S^\mu{}_{\lambda\rho} = -i\frac{\partial \mathcal{L}}{\partial \partial_\mu A_\sigma}(S_{\lambda\rho})^\sigma{}_\nu A^\nu = F^\mu{}_\rho A_\lambda - F^\mu{}_\lambda A_\rho$.

For the spin $\frac{1}{2}$ Dirac case: $S^\mu{}_{\lambda\rho} = \bar{\Psi}\gamma^\mu S_{\lambda\rho}\Psi$ get

$$\tilde{T}_{\mu\nu} = \frac{i}{2}\left[\bar{\Psi}\gamma_\mu\partial_\nu\Psi + \bar{\Psi}\gamma_\nu\partial_\mu\Psi - (\partial_\nu\bar{\Psi})\gamma_\mu\Psi - (\partial_\mu\bar{\Psi})\gamma_\nu\Psi\right]$$

End of QFT course notes Sep 1 2018.