Problems on Probability

Science Academies' Refresher Course on Theoretical Physics
15 June - 1 July, 2023 at Bishop Moore College, Mavelikara, Kerala
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1. The 10 volumes of the Landau-Lifshitz series of books are placed at random in a book shelf. What is the probability that they are in proper order $(1-2-3-\cdots)$ from left to right?

2. Draw figures to illustrate the following general relations (i) If $A_1 \subset A_2$ then $\bar{A}_1 \supset \bar{A}_2$. (ii) If $A = A_1 \cup A_2$ then $\bar{A} = \bar{A}_1 \cap \bar{A}_2$ and (c) If $A = A_1 \cap A_2$ then $\bar{A} = \bar{A}_1 \cup \bar{A}_2$.

3. Suppose we are given the relations (i) $AB = A$ and (ii) $A \cup B \cup C = A$. Interpret each relation in words and draw a figure that illustrates it.

4. Consider the rolling of a regular tetrahedron shaped die whose faces are painted with the letters a, b, c, d. The outcome of the roll is defined as the letter that is hidden (face down). We assume that all outcomes are equally likely. (a) What are the elementary events ω and what is the sample space Ω? (b) What is the probability of the event A defined as getting an outcome that is a vowel? (c) Let B be the event defined as getting an outcome that is a consonant. What is its probability? (d) Are A and B mutually exclusive? (e) Are A and B statistically independent events? (f) Define the random variable ξ by $\xi(a) = 1, \xi(b) = 2, \xi(c) = 2, \xi(d) = 2$. Find the probabilities: $\mathbf{P}\{\xi = 1\}$, $\mathbf{P}\{\xi = 2\}$, $\mathbf{P}\{\xi < -1\}$. (g) Find an expression for the cumulative distribution $\Phi_\xi(x)$ and plot it. (h) Find an expression for the probability density function $p_\xi(x)$ and plot it.

5. Suppose ξ is a continuous real random variable with probability density $p_\xi(x) = N/(x^2 + a^2)$ for some constant $a > 0$ and $-\infty < x < \infty$. (a) Find $N(a)$. (b) Plot the probability density for $a = 1$. (c) Find the cumulative distribution function $\Phi_\xi(x)$. (d) Express the probability that ξ is positive ($\mathbf{P}\{0 \leq \xi < \infty\}$) as an integral and find its numerical value.

6. If ξ_1 and ξ_2 are random variables, then show that the variance of their sum is the sum of variances plus a correction term given by twice their covariance
$$\text{var}\,(\xi_1 + \xi_2) = \text{var}\,(\xi_1) + \text{var}\,(\xi_2) + 2\,\text{cov}\,(\xi_1, \xi_2). \tag{1}$$
Argue that the covariance of a pair of independent random variables vanishes so that for independent random variables, the variance of the sum is the sum of variances.

7. What is the variance of a random variable ξ with a uniform distribution on the interval $[a, b]$?

8. What is the characteristic function $f_\xi(t) = \langle e^{i\xi t} \rangle$ of the uniform distribution on the unit interval $[0, 1]$?

9. For a real-valued random variable, derive formulae for the cumulants C_n in terms of the moments G_k for $n = 0, 1, 2, 3$.

10. Show that the generating function of a binomial random variable with parameters n, p is $F_\xi(z) = (pz + q)^n$.

11. Show that the generating function of the Poisson distribution with mean a is

$$F_\xi(z) = \sum_{k=0}^{\infty} P_\xi(k) z^k = e^{a(z-1)}. \tag{2}$$

Use this result to find the first few moments G_1, G_2, G_3, G_4 of the Poisson distribution.

12. Viewing $k = 0, 1, 2, \ldots$, as a parameter, consider the sequence of continuous density functions:

$$p_\eta^{(k)}(s) = \frac{s^k e^{-s}}{k!} \quad \text{for} \quad s \geq 0. \tag{3}$$

Show that $p_\eta^{(k)}(s)$ may be viewed as a probability density function for a continuous positive real random variable η. In other words, show that

$$\int_0^\infty p_\eta^{(k)}(s) ds = \int_0^\infty \frac{s^k e^{-s}}{k!} = 1 \quad \text{for any} \quad k = 0, 1, 2, \ldots. \tag{4}$$

13. Complete the square to show that the characteristic function of the standard Gaussian is the Gaussian

$$f_\xi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{ixt}\, dx = e^{-t^2/2}. \tag{5}$$

14. Suppose ξ is a normal random variable with mean μ and variance σ^2. Find an integral expression for the probability $\mathbf{P}\{|\xi - \mu| < n\sigma\}$ that ξ lies within $n\sigma$ of μ for $n = 1, 2, 3, \cdots$. The numerical values of this probability for $n = 1, 2, 3$ are $0.683, 0.954, 0.997$. So with 99.7 % probability, a gaussian random variable takes values within 3σ of its mean.

15. Show that the probability density of a Brownian random walk in one dimension $p_\xi(x) = (4\pi Dt)^{-1/2} \exp(-x^2/4Dt)$ satisfies the diffusion equation $u_t = D u_{xx}$.