Nonlinear Dynamics, Spring 2020 CMI
Problem set 4
Due at the beginning of lecture on Wed Feb 12, 2020
Vector field on S^1, Bottlenecks, Inhomogeneous linear ODE

1. $\langle \mathbf{14} \rangle$ Consider the vector field on a circle $v(\theta) = \omega - a\sin\theta$ for $\omega, a \geq 0$ describing an overdamped pendulum subject to a constant torque.

   (a) $\langle \mathbf{8} \rangle$ Calculate the time period $T(a;\omega)$ for oscillatory motion (when $a < \omega$) by evaluating the appropriate integral in closed form and comment on its behavior as $a \to \omega^-$, where a bottle-neck forms near $\theta = \pi/2$ in the vicinity of the saddle-node bifurcation.

   (b) $\langle \mathbf{6} \rangle$ Estimate the time $T_{\text{bottleneck}}(r)$ spent near the bottleneck around $x = 0$ for trajectories of the canonical vector field displaying a saddle-node bifurcation $v(x) = r + x^2$ as $r \to 0^+$. Give reasons for any approximation made. Compare the behavior of $T_{\text{bottleneck}}(r)$ with the above time period $T(a;\omega)$ in the appropriate limits.

2. $\langle \mathbf{6} \rangle$ Solve the following inhomogeneous linear ODE for $x(t)$:

$$\dot{x} = \lambda x + y_0 e^{\lambda t} \quad \text{with} \quad x(0) = x_0. \tag{1}$$

   Here y_0 is a constant and $\lambda \neq 0$ is a real constant. Give the intermediate steps in obtaining your solution. Check that it satisfies the initial value problem.