Nonlinear Dynamics, Spring 2020 CMI
Problem set 3
Due at the beginning of lecture on Wednesday Feb 5, 2020
Picard iteration and Bifurcations of 1d vector fields

1. $\langle \mathbf{10} \rangle$ Picard iteration is a method to solve the IVP $\dot{x} = v(x)$, $x(0) = x_0$ and to prove the existence-uniqueness theorem when v is continuously differentiable. It proceeds by defining a sequence of functions $x_k(t)$ for $k = 0, 1, 2, \ldots$ inductively via

$$x_0(t) = x_0 \quad \text{and} \quad x_{k+1}(t) = x_0 + \int_0^t v(x_k(t'))\, dt'. \tag{1}$$

which converge to the solution. Let us illustrate the method with two simple examples.

   (a) $\langle \mathbf{4} \rangle$ Obtain the Picard iterates $x_k(t)$ for $\dot{x} = ax$ subject to the IC $x(0) = x_0$. To which function do they converge as $k \to \infty$?

   (b) $\langle \mathbf{6} \rangle$ Obtain the Picard iterates $\mathbf{x}_k(t)$ for $\dot{\mathbf{x}} = A\mathbf{x}$ where $A = \sigma_1 = (01|10)$ is the first Pauli matrix with IC $\mathbf{x}(0) = (1, 0)^t$ (superscript t denotes transpose). To which $\mathbf{x}(t)$ do they converge as $k \to \infty$?

2. $\langle \mathbf{10} \rangle$ Consider the family of 1D vector fields $\dot{x} = v(x) = \beta \tanh x - x$ with real β a control parameter. Sketch the phase portraits and plot $v(x)$ for three qualitatively different β. Show that it displays a bifurcation (what type of bifurcation is it?) and find the bifurcation point β_c and the fixed point x_* at β_c. Draw a bifurcation diagram and explain how the character of the phase portrait changes.