Mathematical Physics 1: Linear Algebra, CMI

Problem set 4

Instructor: Govind S. Krishnaswami

Due at the beginning of class on Tuesday, August 18.

Matrix of discretized derivative

In the lecture it was mentioned that Newton's equation $\ddot{x} = f$ could be written as a matrix equation when discretized. Here you will do this for the simpler problem of the first derivative. Given the position of a particle $x(t)$, find its (approximate) velocity. We are provided the positions of the particle $x_k \equiv x(t_k)$ at equally spaced times $t_1, t_2, \cdots, t_n$, with $t_{i+1} - t_i = \Delta$.

1. Assemble the positions of the particle in a column vector with n-components X and display it. $< 1 >$

2. The velocity is $\dot{x}(t) = \lim_{\Delta \to 0} \frac{x(t+\Delta) - x(t)}{\Delta}$. Define the approximate velocity $\dot{x}_k$ at any time t_k as the difference quotient with $\Delta = 1$. Write a formula for $\dot{x}_k$. You may assume that the particle returned to its original position at the end of the journey $x(t_{n+1}) = x(t_1)$. $< 1 >$

3. List out $\dot{x}_k$ for $k = 1, 2, 3, n-1, n$. $< 2 >$

   The approximate velocities are assembled in a column vector $V = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix}$

4. Find the matrix D, which when applied to the column of positions, produces the column of approximate velocities $V = DX$. $< 2 >$

5. Write out the matrix D_n for the case $n = 4$ explicitly. $< 1 >$

6. What vector space do V and X live in? $< 1 >$

7. Is D_4 upper triangular? Is D_4 symmetric? $< 1 >$

8. Using elementary row operations, bring D_4 to row echelon form. $< 2 >$

9. What is the rank of D_4? $< 1 >$

10. What is the determinant of D_4? Is it invertible? $< 2 >$

11. Find a column vector annihilated by D_4. If there is a non-zero vector in the kernel of D_4, find it, otherwise explain why there isn't one. (Hint: Use multiplication by columns to think of DX or use Gaussian elimination to solve for the kernel.) $< 2 >$

12. What sort of physical motion does the above-discovered vector in the kernel represent? $< 2 >$

13. Can you guess all vectors in the kernel of D_n and their physical meaning? $< 2 >$