Mathematical Physics 1: Linear Algebra, CMI

Problem set 2

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Due at the beginning of class on Tuesday 11 August.

Matrix multiplication and Pauli Matrices.

The Pauli matrices are the 2×2 matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

They are important in quantum mechanics and group theory. Here $i = \sqrt{-1}$ is the imaginary unit with $i^2 = -1$. $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the 2×2 identity matrix.

1. Calculate σ_1^2, multiplying rows by columns (dot products).
2. Calculate σ_2^2, multiplying by rows (linear combination of rows of right member of product)
3. Calculate σ_3^2 multiplying by columns (linear combination of columns of left member of product)
4. Calculate $\sigma_1\sigma_2$ multiplying columns by rows (sum of outer products). Express the answer in terms of the Pauli matrices.
5. Calculate $\sigma_2\sigma_3$ multiplying by columns. Express the answer in terms of the Pauli matrices..
6. Calculate $\sigma_3\sigma_1$ multiplying by rows. Express the answer in terms of the Pauli matrices.
7. δ_{ij} for $1 \leq i,j \leq n$ is the Kronecker delta, it vanishes for $i \neq j$ and equals 1 for $i = j$. Which matrix are δ_{ij} the entries of? Write δ_{ij} as a matrix for $n = 2,3$
8. For $1 \leq i,j \leq 3$, ϵ_{ijk} is the Levi-Civita symbol (epsilon tensor). $\epsilon_{123} = 1$ and it is antisymmetric under the interchange of any two neighbouring indices, such as $\epsilon_{ijk} = -\epsilon_{jik}$. Find ϵ_{ijk} for all possible values of $1 \leq i,j,k \leq 3$.
9. Using these results, verify that the products of the Pauli matrices can be summarized in the formula

$$\sigma_a\sigma_b = \delta_{ab}I + i\epsilon_{abc}\sigma_c, \quad \text{where} \quad a,b = 1,2,3. \tag{2}$$

   The repeated index c is summed from 1 to 3.

10. The commutator of a pair of matrices measures to what extent $AB \neq BA$. More precisely, $[A,B] = AB - BA$. Using the above results, find $[\sigma_1,\sigma_2], [\sigma_2,\sigma_3], [\sigma_3,\sigma_1]$ and express the answers in terms of the Pauli matrices. The final answer should fit in one line.