Introduction to Fluid Mechanics

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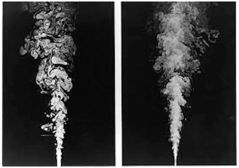


Acknowledgements

- Let me begin by thanking the organizers at Alva's college for inviting me to speak on Fluid Mechanics at this workshop.
- The joint work presented towards the end of this lecture was done with my collaborator A Thyagaraja and graduate students Sachin Phatak and Sonakshi Sachdev. I have learned much from them, as well as from teachers (including my PhD supervisor S G Rajeev) and colleagues. I will mention some reference books as we go along.
- Though I have not yet mentioned the individual sources, I would like to thank the authors for the illustrations in these slides [most are scanned from books mentioned or downloaded from the internet (Wikipedia, Google images)].
- Special thanks are due to Sonakshi Sachdev for her help in preparing these slides.

Continuum, Fluid element, Local thermal equilibrium

- In fluid mechanics we are not interested in positions and velocities of individual molecules. Focus instead on fluid variables, which are quantities like velocity, pressure, density and temperature that we can assign to a fluid element by averaging over it.
- By a fluid element, we mean a sufficiently large collection of molecules so that concepts such as 'volume occupied' make sense and yet small by macroscopic standards so that the velocity, density, pressure etc. are roughly constant over its extent. E.g.: divide a container with 10^{23} molecules into 10000 cells, each containing 10^{19} molecules.
- A flowing fluid is not in global thermal equilibrium. Collisions establish local thermodynamic equilibrium so that we can assign a local $T, p, \rho, E, \ldots$ to fluid elements, satisfying the laws of thermodynamics.
- Fluid description applies to phenomena on length-scale $\gg$ mean free path. On shorter length-scales, fluid description breaks down, but kinetic theory (Boltzmann transport equation) applies.

Eulerian and Lagrangian viewpoints

- In the Eulerian description, we are interested in the time development of fluid variables at a given point of observation $\vec{r} = (x, y, z)$. Interesting if we want to know how density changes, say, above my head. However, different fluid particles will arrive at the point $\vec{r}$ as time elapses.
- It is also of interest to know how the corresponding fluid variables evolve, not at a fixed location but for a fixed fluid element, as in a Lagrangian description.
- This is especially important since Newton's second law applies directly to fluid particles, not to point of observation!
- So we ask how a variable changes along the flow, so that the observer is always attached to a fixed fluid element.

Leonhard Euler and Joseph Louis Lagrange

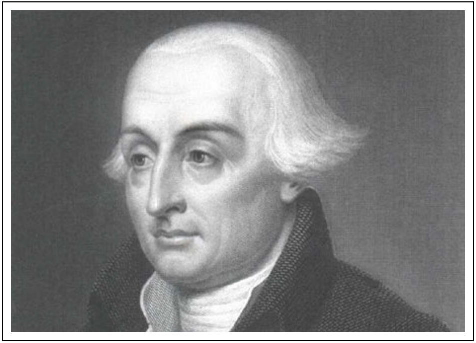


Figure: Leonhard Euler (left) and Joseph Louis Lagrange (right).

Material derivative measures rate of change along flow

- Change in density of a fluid element in time dt as it moves from $\mathbf{r}$ to $\mathbf{r}+d\mathbf{r}$ is

$$d\rho = \rho(\mathbf{r}+d\mathbf{r}, t+dt) - \rho(\mathbf{r},t) \approx \frac{\partial \rho}{\partial t} dt + d\mathbf{r}\cdot\nabla\rho. \tag{1}$$

- Divide by dt, let $dt \to 0$ and use $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ to get instantaneous rate of change of density of a fluid element located at $\mathbf{r}$ at time t:

$$\frac{D\rho}{Dt} \equiv \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v}\cdot\nabla\rho. \tag{2}$$

- Material derivative of any quantity s in a flow field $\mathbf{v}$ is defined as $\frac{Ds}{Dt} = \partial_t s + \mathbf{v}\cdot\nabla s$. It measures rate of change of s in a fluid element as the element moves around.
- Material derivative of velocity $\frac{D\mathbf{v}}{Dt} = \partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}$ gives the instantaneous acceleration of a fluid element with velocity $\mathbf{v}$ located at $\mathbf{r}$ at time t.
- As a 1^{st} order differential operator it satisfies Leibnitz' product rule

$$\frac{D(fg)}{Dt} = f\frac{Dg}{Dt} + g\frac{Df}{Dt}, \qquad \frac{D(\rho\mathbf{v})}{Dt} = \rho\frac{D\mathbf{v}}{Dt} + \mathbf{v}\frac{D\rho}{Dt} \tag{3}$$

Continuity equation and incompressibility

- Rate of increase of mass in a fixed vol V is equal to the influx of mass.

$$\frac{d}{dt}\int_V \rho d\mathbf{r} = -\int_{\partial V} \rho \mathbf{v}\cdot\hat{n}\, dS = -\int_V \nabla\cdot(\rho\mathbf{v})\, d\mathbf{r} \quad \Rightarrow \int_V [\rho_t + \nabla\cdot(\rho\mathbf{v})]\, d\mathbf{r} = 0.$$

- As V is arbitrary, we get continuity equation for local mass conservation:

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0 \qquad \text{or} \qquad \partial_t\rho + \mathbf{v}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{v} = 0. \tag{4}$$

- In terms of material derivative, $\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0$.
- Flow is incompressible if $\frac{D\rho}{Dt} = 0$: density of a fluid element is constant. Since mass of a fluid element is constant, incompressible flow preserves volume of fluid element.
- Flow is incompressible if $\nabla\cdot\mathbf{v} = 0$, i.e., $\mathbf{v}$ is divergence-free or solenoidal.
- $\nabla\cdot\mathbf{v} = \lim_{V,\delta t\to 0} \frac{1}{\delta t}\frac{\delta V}{V}$ measures fractional rate of change of volume of a small fluid element.
- Most important incompressible flow is constant ρ in space and time.
- Incompressibility is a property of the flow and not just the fluid material!

Sound speed, Mach number

- Flow may be approximated as incompressible in regions where flow speed is small compared to local sound speed $c_s = \sqrt{\frac{\partial p}{\partial \rho}} \sim \sqrt{\gamma p/\rho}$ for adiabatic flow of an ideal gas with $\gamma = C_p/C_v$.
- Recall that compressibility $\beta = \frac{\partial \rho}{\partial p}$ measures how much density can be increased by increasing pressure. Incompressible fluid has $\beta = 0$, so $c^2 = 1/\beta = \infty$. In practice, an approximately incompressible fluid is one with very large sound speed (much more than flow speeds).
- Common flows in water are incompressible. So study of incompressible flow is called hydrodynamics.
- High speed flows in air/gases tend to be compressible. So compressible flow is called aerodynamics or gas dynamics.
- Incompressible hydrodynamics may be derived from compressible gas dynamic equations in the limit of small Mach number $M = |\mathbf{v}|/c_s \ll 1$.
- At high Mach numbers $M \gg 1$ we have super-sonic flow and phenomena like shocks.

Newton's 2^{nd} law for fluid element: Inviscid Euler equation

- Consider a fluid element of volume δV. Mass $\times$ acceleration is $\rho(\delta V)\frac{D\mathbf{v}}{Dt}$.
- Force on fluid element includes 'body force' like gravity derived from a potential ϕ. E.g. $\mathbf{F} = -\rho(\delta V)\nabla\phi$ where $-\nabla\phi$ is acceleration due to gravity.
- Also have surface force on a volume element, due to pressure exerted on it by neighbouring elements

$$F_{\text{surface}} = -\int_{\partial V} p\,\hat{n}\,dS = -\int_V \nabla p dV; \quad \text{if} \quad V = \delta V \quad \text{then} \quad F_{\text{surf}} \approx -\nabla p(\delta V).$$

- Newton's 2^{nd} law then gives the celebrated (inviscid) Euler equation

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{\nabla p}{\rho} - \nabla\phi; \qquad \mathbf{v}\cdot\nabla\mathbf{v} \to \text{'advection term'} \tag{5}$$

- Continuity & Euler eqns. are first order in time derivatives: to solve initial value problem, must specify $\rho(\mathbf{r}, t=0)$ and $\mathbf{v}(\mathbf{r}, t=0)$.
- **Boundary conditions:** Euler equation is 1st order in space derivatives; impose BC on $\mathbf{v}$, not $\partial_i\mathbf{v}$. On solid boundaries normal component of velocity vanishes $\mathbf{v}\cdot\hat{n} = 0$. As $|\mathbf{r}| \to \infty$, typically $\mathbf{v} \to 0$ and $\rho \to \rho_0$.

Isaac Newton

Figure: Isaac Newton

Barotropic flow and specific enthalpy

- Euler & continuity are 4 eqns for 5 unknowns $\rho, \mathbf{v}, p$. Need another eqn.
- In local thermodynamic equilibrium, pressure may be expressed as a function of density and entropy. For isentropic flow it reduces to a barotropic relation $p = p(\rho)$. It eliminates p and closes the system of equations. E.g. $p \propto \rho^\gamma$ adiabatic flow of ideal gas; $p \propto \rho$ for isothermal.
- In barotropic flow, $\nabla p/\rho$ can be written as the gradient of an 'enthalpy'

$$h(\rho) = \int_{\rho_0}^{\rho} \frac{p'(\tilde\rho)}{\tilde\rho} d\tilde\rho \quad \Rightarrow \quad \nabla h = h'(\rho)\nabla\rho = \frac{p'(\rho)}{\rho}\nabla\rho = \frac{\nabla p}{\rho}. \tag{6}$$

- Reason for the name enthalpy: 1^{st} law of thermodynamics $dU = TdS - pdV$ becomes $dH = TdS + Vdp$ for enthalpy $H = U + pV$. For an isentropic process $dS = 0$, so $dH = Vdp$.
- Dividing by mass of fluid M we get $d(H/M) = (V/M)dp$. Defining enthalpy per unit mass $h = H/M$ and density $\rho = M/V$ gives $dh = dp/\rho$.

Barotropic flow and conserved energy

- In barotropic flow $p = p(\rho)$ and $\nabla p/\rho$ is gradient of enthalpy ∇h. So the Euler equation becomes

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla h. \tag{7}$$

- Using the vector identity $\mathbf{v}\cdot\nabla\mathbf{v} = \nabla(\frac{1}{2}\mathbf{v}^2) + (\nabla\times\mathbf{v})\times\mathbf{v}$, we get

$$\partial_t \mathbf{v} + (\nabla\times\mathbf{v})\times\mathbf{v} = -\nabla\left(h + \frac{1}{2}\mathbf{v}^2\right) \quad \text{where} \quad \nabla h = \frac{1}{\rho}\nabla p. \tag{8}$$

- Barotropic flow has a conserved energy: kinetic + compressional

$$E = \int\left[\frac{1}{2}\rho\mathbf{v}^2 + U(\rho)\right]d^3r, \quad \text{where} \quad U'(\rho) = h(\rho). \tag{9}$$

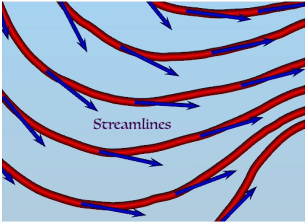
  $U = p/(\gamma-1)$ for adiabatic flow of ideal gas. For monatomic ideal gas $\gamma = 5/3$ and compressional energy is $(3/2)pV = (3/2)NkT$.

- More generally, the Euler and continuity equations are supplemented by an equation of state and energy equation (1^{st} law of thermodynamics).

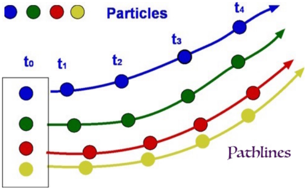
Flow visualization: Stream-, Streak- and Path-lines

- If $\mathbf{v}(\mathbf{r},t) = \mathbf{v}(\mathbf{r})$ is time-independent everywhere, the flow is steady.
- Stream, streak and pathlines coincide for steady flow. They are the integral curves (field lines) of $\mathbf{v}$, everywhere tangent to $\mathbf{v}$:

$$\frac{d\mathbf{r}}{ds} = \mathbf{v}(\mathbf{r}(s)) \text{ or } \frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z}; \quad \mathbf{r}(s_o) = \mathbf{r}_o.$$

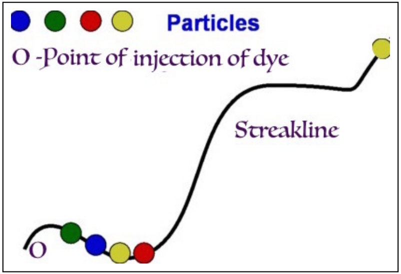


- In unsteady flow, streamlines at time t_0 encode the instantaneous velocity pattern. Streamlines at a given time do not intersect.
- Path-lines are trajectories of individual fluid 'particles' (e.g. speck of dust stuck to fluid). At a point P on a path-line, it is tangent to $\mathbf{v}(P)$ at the time the particle passed through P. Pathlines can (self)intersect at $t_1 \neq t_2$.



Streak-lines

- Streak-line: Dye is continuously injected into a flow at a fixed point P. Dye particle sticks to the first fluid particle it encounters and flows with it. Resulting high-lighted curve is the streak-line through P. So at a given time of observation t_{obs}, a streak-line is the locus of all current locations of particles that passed through P at some time $t \le t_{\text{obs}}$ in the past.

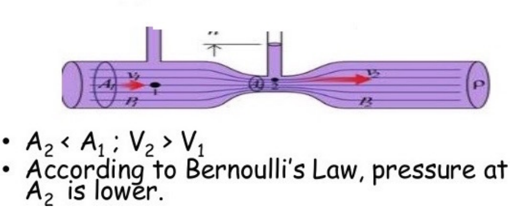


Steady Bernoulli principle

- Euler's equation for barotropic flow subject to a conservative body force potential Φ (e.g. $\Phi = gz$ for gravity at height z) is

$$\frac{\partial \mathbf{v}}{\partial t} + (\nabla \times \mathbf{v}) \times \mathbf{v} = -\nabla \mathcal{B} \quad \text{where} \quad \mathcal{B} = \frac{1}{2}\mathbf{v}^2 + h + \Phi \tag{10}$$

- For steady flow $\partial_t \mathbf{v} = 0$. Dotting with $\mathbf{v}$ we find the Bernoulli specific energy $\mathcal{B}$ is constant along streamlines: $\mathbf{v} \cdot \nabla \mathcal{B} = 0$.
- For incompressible (constant density) flow, enthalpy $h = p/\rho$. Thus along a streamline $\frac{1}{2}\mathbf{v}^2 + p/\rho + gz$ is constant. For roughly horizontal flow, pressure is lower where velocity is higher.
- E.g. Pressure drops as flow speeds up at constrictions in a pipe. Try to separate two sheets of paper by blowing air between them!



Daniel Bernoulli

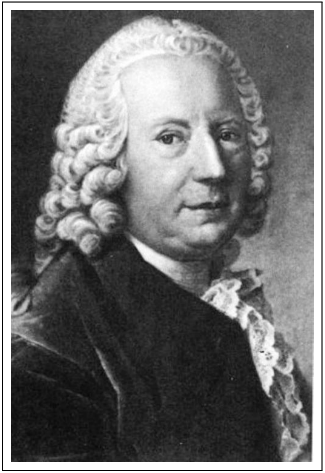
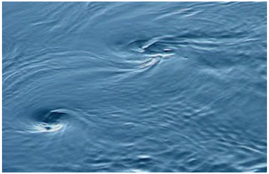


Figure: Daniel Bernoulli

Vorticity and circulation

- *Vorticity* $\mathbf{w} = \nabla \times \mathbf{v}$ is a measure of local *rotation/angular momentum* in a flow. A flow without vorticity is called *irrotational*.
- Eddies and vortices are manifestations of vorticity in a flow. $[\mathbf{w}] = 1/T$, a frequecy.
- Given a closed contour C in a fluid, the circulation around the contour $\Gamma(C) = \oint_C \mathbf{v} \cdot d\mathbf{l}$ measures how much $\mathbf{v}$ 'goes round' C. By Stokes' theorem, it equals the flux of vorticity across a surface that spans C.

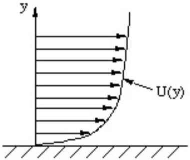


$$\Gamma(C) = \oint_C \mathbf{v} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{v}) \cdot dS = \int_S \mathbf{w} \cdot dS \quad \text{where} \quad \partial S = C.$$

- Enstrophy $\int \mathbf{w}^2 \, d\mathbf{r}$ measures global vorticity. It is conserved in ideal 2d flows, but not in 3d, and could diverge due to 'vortex stretching'.

Examples of flow with vorticity $\mathbf{w} = \nabla \times \mathbf{v}$

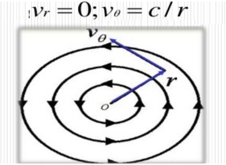
- Shear flow with horizontal streamlines is an example of flow with vorticity: $\mathbf{v}(x,y,z) = (U(y),0,0)$. Vorticity $\mathbf{w} = \nabla \times \mathbf{v} = -U'(y)\hat{z}$.



- A bucket of fluid rigidly rotating at small angular velocity $\Omega\hat{z}$ has $\mathbf{v}(r,\theta,z) = \Omega\hat{z} \times \mathbf{r} = \Omega r\hat{\theta}$. The corresponding vorticity $\mathbf{w} = \nabla \times \mathbf{v} = \frac{1}{r}\partial_r(r v_\theta)\hat{z}$ is constant over the bucket, $\mathbf{w} = 2\Omega\hat{z}$.

- The planar azimuthal velocity profile $\mathbf{v}(r,\theta) = \frac{c}{r}\hat{\theta}$ has circular streamlines. It has no vorticity $\mathbf{w} = \frac{1}{r}\partial_r(r\frac{c}{r})\hat{z} = 0$ except at $r = 0$: $\mathbf{w} = 2\pi c\delta^2(\mathbf{r})\hat{z}$. The circulation around any contour enclosing the origin is a *nonzero* constant

$$\oint \mathbf{v} \cdot d\mathbf{l} = \oint (c/r) r\, d\theta = 2\pi c$$



Freezing of $\mathbf{w}$ into $\mathbf{v}$: Kelvin & Helmholtz theorems

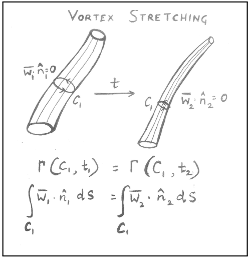
- Taking the curl of the Euler equation $\partial_t \mathbf{v} + (\nabla \times \mathbf{v}) \times \mathbf{v} = -\nabla\left(h + \frac{1}{2}\mathbf{v}^2\right)$ allows us to eliminate the pressure term in barotropic flow to get

$$\partial_t \mathbf{w} + \nabla \times (\mathbf{w} \times \mathbf{v}) = 0. \tag{11}$$

- This may be interpreted as saying that vorticity is 'frozen' into $\mathbf{v}$.
- The flux of $\mathbf{w}$ through a surface moving with the flow is constant in time:

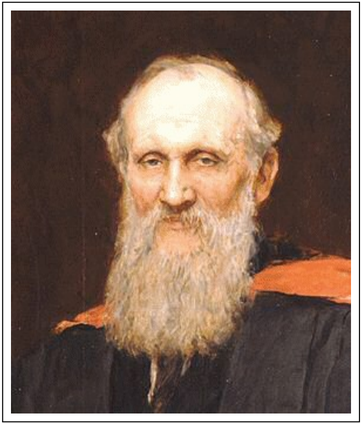
$$\frac{d}{dt}\int_{S_t} \mathbf{w} \cdot dS = 0 \quad \text{or} \quad \frac{d}{dt}\oint_{C_t} \mathbf{v} \cdot d\mathbf{l} = \frac{d\Gamma}{dt} = 0 \quad \text{where} \quad C_t = \partial S_t.$$

- Kelvin's theorem: circulation around a *material* contour is constant. In particular, in the absence of viscosity, eddies and vortices cannot develop in an initially irrotational flow.



- Vortex tubes are cylindrical surfaces everywhere tangent to $\mathbf{w}$. Flow takes vortex tubes to vortex tubes but tends to stretch and bend them. Γ is indep. of time and choice of encircling contour.

Lord Kelvin and Hermann von Helmholtz



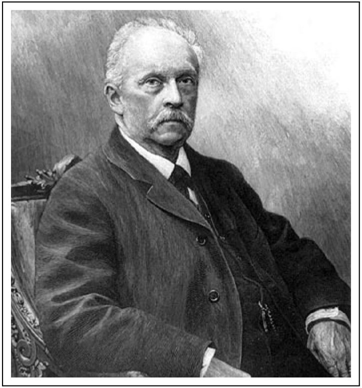
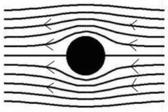


Figure: Lord Kelvin (left) and Hermann von Helmholtz (right).

Irrotational incompressible inviscid flow around cylinder

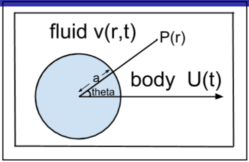


- When flow is irrotational ($\mathbf{w} = \nabla \times \mathbf{v} = 0$) we may write $\mathbf{v} = -\nabla\phi$. Velocity potential ϕ is like the electrostatic potential for $\nabla \times \mathbf{E} = 0$.
- Incompressibility $\nabla \cdot \mathbf{v} = 0 \Rightarrow \phi$ satisfies Laplace's equation $\nabla^2\phi = 0$.
- We impose impenetrable boundary conditions: normal component of velocity vanishes on solid surfaces: $\frac{\partial \phi}{\partial \hat{n}} = 0$ on boundary (Neumann BC).
- Consider flow with asymptotic velocity $-U\hat{x}$ past a fixed infinite cylinder of radius a with axis along $\hat{z}$. Due to translation invariance along z, this is a 2d problem in the r, θ plane.
- The BCs are $\frac{\partial \phi}{\partial r} = 0$ at $r = a$ and $\phi \to Ur\cos\theta$ as $r \to \infty$ (so $\mathbf{v} \to -U\hat{x}$).
- Separating variables, gen. soln. to $\nabla^2\phi = (1/r)\partial_r(r\partial_r\phi) + (1/r^2)\partial_\theta^2\phi = 0$ is

$$\phi = (A_0 + B_0 \ln r) + \sum_{n=1}^{\infty}\left(A_n r^n + \frac{B_n}{r^n}\right)(C_n \cos n\theta + D_n \sin n\theta). \qquad (12)$$

- Imposing BCs we get $\phi = U\cos\theta\left(r + \frac{a^2}{r}\right)$. The corresponding velocity field is $\mathbf{v} = -\nabla\phi = -U\hat{x} + U\frac{a^2}{r^2}(\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$.

Potential flow and the added mass effect



- Velocity field for potential flow past a cylinder is $\mathbf{v} = -U\hat{x} + U\frac{a^2}{r^2}(\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$.
- Now consider problem of a cylinder moving with velocity $U\hat{x}$ through a fluid asymptotically at rest.
- By a Galilean transformation, the velocity field around the cylinder is $\mathbf{v}' = \mathbf{v} + U\hat{x} = U\frac{a^2}{r'^2}(\cos\theta'\,\hat{r}' + \sin\theta'\,\hat{\theta}')$ where r', θ' are relative to the center of the cylinder.
- This example can be used to illustrate the added mass effect. The force required to accelerate a body (of mass M at $\dot{U}$) through potential flow exceeds $M\dot{U}$, since part of the force applied goes to accelerate the fluid.
- Indeed the flow KE $\frac{1}{2}\rho\iint_a^\infty(\mathbf{v}')^2 r'\,d\theta'\,dr' = \frac{1}{2}\rho\pi a^2 U^2 \equiv \frac{1}{2}M'U^2$ is quadratic in U just like the KE of cylinder itself. Thus $K_{\text{total}} = \frac{1}{2}(M+M')U^2$.
- The associated power to be supplied is $\dot{K}_{\text{total}} = F \cdot U$. So a force $F = (M+M')\dot{U}$ is required to accelerate the body at $\dot{U}$. Body behaves as if it has an effective mass $M+M'$. M' is its added or virtual mass. Ships must carry more fuel than expected after accounting for viscosity.

Sound waves in compressible flow

- Sound waves are excitations of the ρ or p fields. Arise in compressible flows, where regions of compression and rarefaction can form.
- To derive the simplest equation for sound waves we linearize the continuity and Euler eqns around the static solution $\mathbf{v} = 0$ and $\rho = \rho_0$:

$$\mathbf{v} = 0 + \mathbf{v}_1(\mathbf{r}, t), \quad \rho = \rho_0 + \rho_1(\mathbf{r}, t) \quad \text{and} \quad p = p_0 + p_1(\mathbf{r}, t). \tag{13}$$

  where $\mathbf{v}_1, \rho_1$ and p_1 are assumed small (treated to linear order).
- Ignoring quadratic terms in small quantities, the continuity eqn. $\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$ becomes $\partial_t \rho_1 + \rho_0 \nabla \cdot \mathbf{v}_1 = 0$.
- Assuming pressure variations are linear in density variations $p_1 = c^2 \rho_1$, the Euler eqn $\rho(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla p$ becomes $\rho_0 \partial_t \mathbf{v}_1 = -c^2 \nabla \rho_1$ or upon taking a divergence, $\rho_0 \partial_t (\nabla \cdot \mathbf{v}_1) = -c^2 \nabla^2 \rho_1$.
- Eliminating $\nabla \cdot \mathbf{v}_1$ using continuity eqn we get the wave equation for density variations $\partial_t^2 \rho_1 = c^2 \nabla^2 \rho_1$. We identify c as the sound speed.
- For incompressible flow, the sound speed is infinite: $c^2 = \frac{\delta p}{\delta \rho} \to \infty$ as the density variation is vanishingly small even for large pressure variations.

Heat diffusion equation

- Empirically it is found that the heat flux between bodies grows with the temperature difference. Fourier's law of heat diffusion states that the heat flux density vector (energy crossing unit area per unit time) is proportional to the negative gradient in temperature

$$\mathbf{q} = -k\nabla T \quad \text{where} \quad k = \text{thermal conductivity}. \tag{14}$$

- Consider gas in a fixed volume V. The increase in internal energy $U = \int_V \rho c_v T d\mathbf{r}$ must be due to the influx of heat across its surface S.

$$\int_V \partial_t(\rho c_v T) d\mathbf{r} = \int_S k\nabla T \cdot \hat{n}\, dS = k \int_V \nabla \cdot \nabla T\, d\mathbf{r}. \tag{15}$$

c_v = specific heat/mass (at constant volume, no work) and ρ = density.

- V is arbitrary, so integrands must be equal. Heat equation follows:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \quad \text{where} \quad \alpha = \frac{k}{\rho c_v} \quad \text{is thermal diffusivity}. \tag{16}$$

- Heat diffusion is dissipative, temperature differences even out and heat flow stops at equilibrium temperature. It is not time-reversible.

Including viscosity: Navier-Stokes equation

- Heat equation $\partial_t T = \alpha \nabla^2 T$ describes diffusion from hot → cold regions.
- (Shear) viscosity causes diffusion of velocity from a fast layer to a neighbouring slow layer of fluid. The viscous stress is $\propto$ velocity gradient. If a fluid is stirred and left, viscosity brings it to rest.
- By analogy with heat diffusion, velocity diffusion is described by $\nu \nabla^2 \mathbf{v}$.
- Kinematic viscosity ν has dimensions of diffusivity (areal velocity L^2/T).
- *Postulate* the Navier-Stokes equation for viscous *incompressible* flow:

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v} \qquad \text{(NS)}. \tag{17}$$

- NS has not been derived from molecular dynamics except for dilute gases. It is the simplest equation consistent with physical requirements and symmetries. It's validity is restricted by experiment.
- NS is second order in space derivatives unlike the inviscid Euler eqn. Experimentally relevant boundary condition is 'no-slip' at solid surfaces.

Claude Louis Navier, Saint Venant and George Stokes

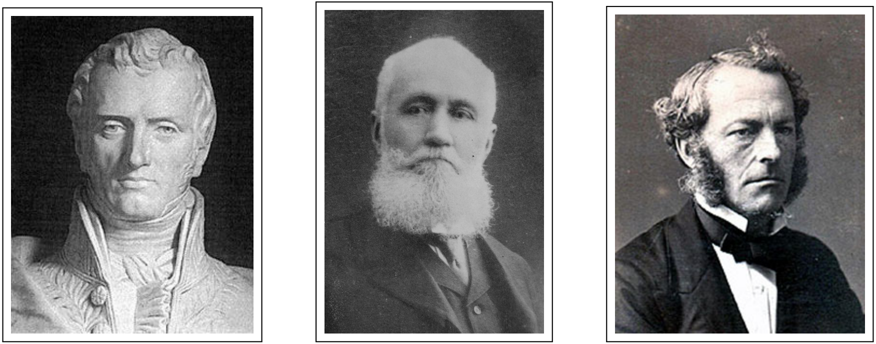


Figure: Claude Louis Navier (left), Saint Venant (middle) and George Gabriel Stokes (right).

Reynolds number $\mathcal{R}$ and similarity principle

- Incompressible flows with same $\mathcal{R}$ have similar (rescaled) flow patterns.
- Suppose U and L are a typical speed and length associated to a flow (e.g. asymptotic flow speed U past a sphere of size L). Define

$$\text{dim. less variables} \quad \mathbf{r}' = \frac{\mathbf{r}}{L}, \;\; \nabla' = L\nabla \;\; t' = \frac{U}{L}t, \;\; \mathbf{v}' = \frac{\mathbf{v}}{U}, \;\; \mathbf{w}' = \frac{\mathbf{w}L}{U}.$$

- Then incompressible NS vorticity eqn in non-dimensional variables is

$$\frac{\partial \mathbf{w}'}{\partial t'} + \nabla' \times (\mathbf{w}' \times \mathbf{v}') = \frac{\nu}{LU} \nabla'^2 \mathbf{w}'. \quad \text{We define } \frac{1}{\mathcal{R}} = \frac{\nu}{LU}. \tag{18}$$

- ν enters only through $\mathcal{R}$. If 2 flows expressed in scaled variables have same $\mathcal{R}$ and BCs, then flow patterns are similar. Flow around aircraft is simulated in wind tunnel using a scaled down aircraft with same $\mathcal{R}$.
- $\mathcal{R}$ is a measure of ratio of inertial to viscous forces

$$\frac{F_{\text{inertial}}}{F_{\text{viscous}}} = \frac{|\mathbf{v} \cdot \nabla \mathbf{v}|}{|\nu \nabla^2 \mathbf{v}|} \sim \frac{U^2/L}{\nu U/L^2} \sim \frac{LU}{\nu} = \mathcal{R}. \tag{19}$$

- When $\mathcal{R}$ is small (e.g. in slow creeping flow), viscous forces dominate inertial forces and vice versa.

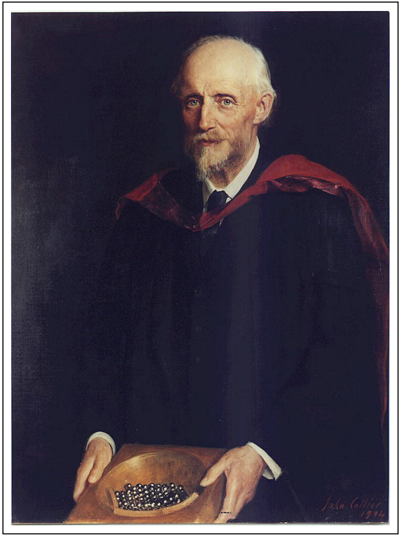


Figure: Osbourne Reynolds

Stokes flow: drag on a sphere in steady creeping flow

- Stokes studied incompressible (constant ρ) flow around a sphere of radius a moving through a viscous fluid with velocity $\mathbf{U}$

$$\mathbf{v}'_t + \mathbf{v}' \cdot \nabla' \mathbf{v}' = -\frac{1}{\rho} \nabla' p + \frac{1}{\mathcal{R}} \nabla'^2 \mathbf{v}, \quad \frac{1}{\mathcal{R}} = \frac{\nu}{aU} \tag{20}$$

- For steady flow $\partial_t \mathbf{v}' = 0$. For creeping flow ($\mathcal{R} \ll 1$) we may ignore advection term and take a curl to eliminate pressure to get

$$\nabla'^2 \mathbf{w}' = 0. \tag{21}$$

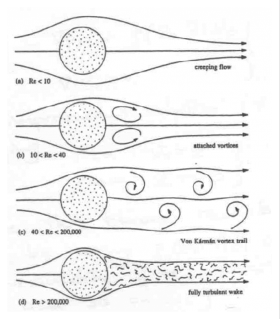
- By integrating the stress over the surface Stokes found the drag force

$$F_i = -\int \sigma_{ij} n_j dS \quad \Rightarrow \quad \mathbf{F}_{\text{drag}} = -6\pi\rho\nu a \mathbf{U}. \tag{22}$$

- Upto 6π factor, this follows from dimensional analysis! Magnitude of drag force is $F_D = \frac{12}{\mathcal{R}} \times \frac{1}{2}\pi a^2 U^2$. $12/\mathcal{R}$ is the drag coefficient for stokes flow.

Drag on a sphere at higher Reynolds number

- At higher speeds ($\mathcal{R} \gg 1$), naively expect viscous term to be negligible. However, experimental flow is far from ideal (inviscid) flow!
- At higher $\mathcal{R}$, flow becomes unsteady, vortices develop downstream and eventually a turbulent wake is generated.



- Dimensional analysis implies drag force on a sphere is expressible as $F_D = \frac{1}{2} C_D(\mathcal{R})\, \pi a^2 \rho U^2$, where $C_D = C_D(\mathcal{R})$ is the dimensionless drag coefficient, determined by NS equation.
- Comparing with Stokes' formula for creeping flow at $\mathcal{R} \ll 1$ we must have $C_D \sim 12/\mathcal{R}$ as $\mathcal{R} \to 0$.
- Significant experimental deviations from Stokes' law: enhancement of drag at higher $1 \leq \mathcal{R} \leq 10^5$, then drag drops with increasing U!

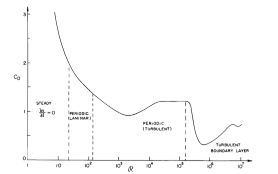
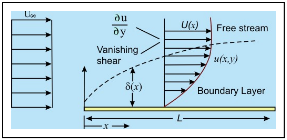


Fig. 41-4. The drag coefficient C_D of a circular cylinder as a function of the Reynolds number.

Drag crisis clarified by Prandtl's boundary layers

- In inviscid flow (Euler equation) tangential velocity on solid surfaces is unconstrained, can be large.
- For viscous NS flow, no slip BC implies tangential $\mathbf{v} = 0$ on solid surfaces.
- Even for low viscosity, there is a thin boundary layer where tangential velocity drops rapidly to zero. In the boundary layer, cannot ignore $\nu\nabla^2\mathbf{v}$.



- Though upstream flow is irrotational, vortices are generated in the boundary layer due to viscosity. These vortices are carried downstream in a (turbulent) wake.
- Larger vortices break into smaller ones and so on, due to inertial forces. Small vortices (at the Taylor microscale) dissipate energy due to viscosity increasing the drag for moderate $\mathcal{R}$.

Ludwig Prandtl

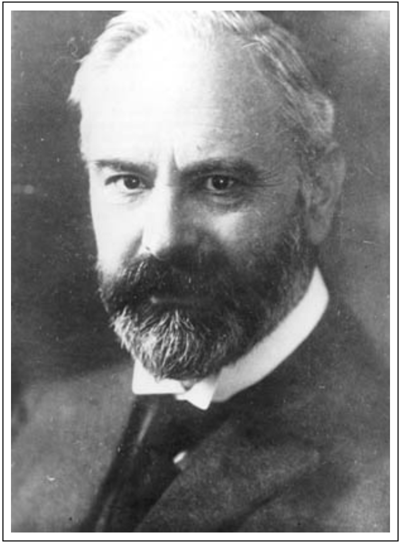
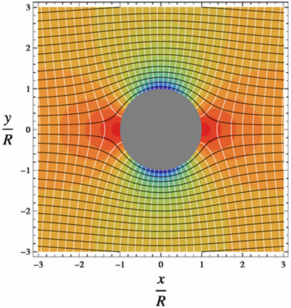


Figure: Ludwig Prandtl

2D Incompressible flows: stream function

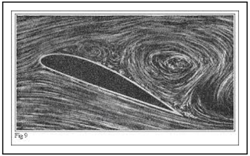
- In 2D incompressible flow the velocity components are expressible as derivatives of a stream function: $\mathbf{v} = (u, v) = (-\psi_y, \psi_x)$. Incompressibility condition $\nabla \cdot \mathbf{v} = -\psi_{yx} + \psi_{xy} = 0$ is identically satisfied.
- Streamlines defined by $\frac{dx}{u} = \frac{dy}{v}$ are level curves of ψ. For, along a streamline $d\psi = (\partial_x \psi) dx + (\partial_y \psi) dy = v dx - u dy = 0$.
- If in addition, flow is irrotational ($\mathbf{w} = \nabla \times \mathbf{v} = 0$), then $\mathbf{v}$ admits a velocity potential $\mathbf{v} = -\nabla \phi$ so that $(u, v) = (-\phi_x, -\phi_y)$.

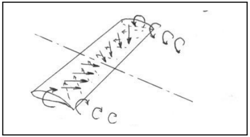


- So ϕ & ψ satisfy the Cauchy Riemann equations: $\phi_x = \psi_y$, $\phi_y = -\psi_x$ and the complex velocity potential $f = \phi + i\psi$ is analytic! ϕ and ψ are harmonic. $\nabla^2 \phi = 0 \Rightarrow$ incompressible and $\mathbf{w} = \nabla^2 \psi \hat{z} = 0 \Rightarrow$ irrotational.
- The level curves of ψ and ϕ are orthogonal : $\nabla \phi \cdot \nabla \psi = -(u, v) \cdot (v, -u) = 0$.
- Complex velocity $u - iv$ is the derivative $-f'(z)$ of the complex potential.

Lift on an airfoil

- Consider an infinite airfoil of uniform cross section. Airflow around it can be treated as 2d.
- Airfoil starts from rest moves left with zero initial circulation. Ignoring $\nu\nabla^2\mathbf{v}$, Kelvin's theorem precludes any circulation developing around wing. Streamlines of potential flow have a singularity as shown in Fig 1.
- Viscosity at rearmost point due to large $\nabla^2\mathbf{v}$ regularizes flow pattern as shown in Fig 2.
- In fact, circulation Γ develops around airfoil. In frame of wing, we have an infinite airfoil with circulation Γ placed perpendicularly in a velocity field v_∞.
- Situation is analogous to infinite wire carrying current I placed perpendicularly in a $\mathbf{B}$ field!



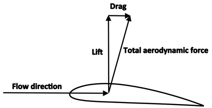


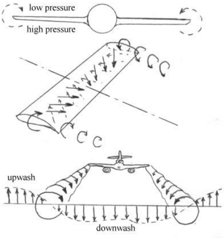
Kutta-Joukowski lift formula for incompressible flow

- Current $\mathbf{j}$ in $\mathbf{B}$ field feels Lorentz force $\mathbf{j}\times\mathbf{B}$ where $\mathbf{j} = \nabla\times\mathbf{B}/\mu_0$ by Ampere. Analogue of Lorentz force is vorticity force in Euler equation

$$\rho\partial_t \mathbf{v} + \rho\mathbf{w}\times\mathbf{v} = -\rho\nabla\sigma + \rho\nu\nabla^2\mathbf{v} \tag{23}$$

- $\mathbf{v}\leftrightarrow\mathbf{B}$, $\mathbf{w}\leftrightarrow\mathbf{j}$. Current carrying wire feels a transverse force BI/unit length. So expect airfoil to feel a force $\rho v_\infty\Gamma$/unit length upwards.
- This force can be obtained using the complex velocity $g = u - iv = v_\infty + a_1/z + \cdots$.
- $\oint g dz = \oint(udx + vdy) + i(udy - vdx) = \oint \mathbf{v}\cdot d\mathbf{l}$ around a contour (streamline) enclosing airfoil just outside boundary layer becomes $\Gamma = 2\pi i a_1$.
- By Bernoulli, force on airfoil $\mathbf{F} = \oint p\,\hat{n}\,dl = -\frac{1}{2}\rho\oint v^2\,\hat{n}\,dl$. Complex force $Z = F_y + iF_x$ can be written as $Z = -(\rho/2)\oint g^2 dz = -\rho v_\infty\Gamma = F_y$.





Nikolay Yegorovich Zhukovsky and Martin Wilhelm Kutta

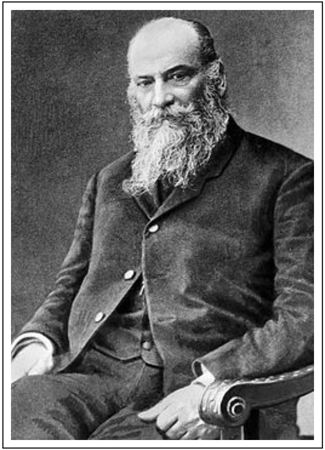
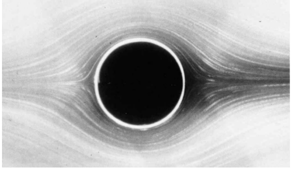
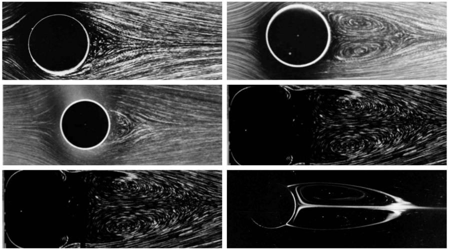


Figure: Nikolay Yegorovich Zhukovsky (left) and Martin Wilhelm Kutta (right).

Transition from laminar to turbulent flow past a cylinder

- Consider flow with asymptotic velocity $U\hat{x}$ past a cylinder of diameter L and axis along $\hat{z}$.
- At very low $\mathcal{R} \approx .16$, the symmetries of the (steady) flow are (a) $y \to -y$ (reflection in $z-x$ plane), (b) time and z translation-invariance (c) left-right ($x \to -x$ and $(u,v,w) \to (u,-v,-w)$).

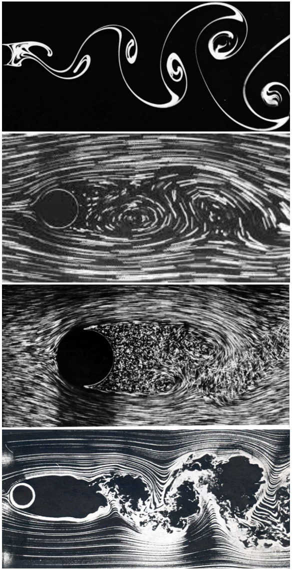




- All these are symmetries of Stokes flow (ignoring the non-linear advection term).
- At $\mathcal{R} \approx 1.5$ a marked left-right asymmetry develops.
- At $\mathcal{R} \approx 5$, change in topology of flow: flow separates and recirculating standing eddies (from diffusion of vorticity) form downstream of cylinder.
- At $\mathcal{R} \approx 40$, flow ceases to be steady, but is periodic in time.

Transition to turbulence in flow past a cylinder

- At $\mathcal{R} \gtrsim 40$, recirculating eddies are periodically (alternatively) shed to form the celebrated von Karman vortex street.
- The z-translation invariance is spontaneously broken when $\mathcal{R} \sim 40 - 75$.
- At higher $\mathcal{R} \sim 200$, flow becomes chaotic with turbulent boundary layer.
- At $\mathcal{R} \sim 1800$, only about two vortices in the von Karman vortex street are distinct before merging into a quasi uniform turbulent wake.
- At much higher $\mathcal{R}$, many of the symmetries of NS are restored in a statistical sense and turbulence is called fully-developed.

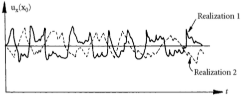


von Karman

Figure: von Karman

What is turbulence? Key features.

- Slow flow or very viscous fluid flow tends to be regular & smooth (laminar). If viscosity is low or speed sufficiently high ($\mathcal{R}$ large enough), irregular/chaotic motion sets in.
- Turbulence is chaos in a driven dissipative system with many degrees of freedom. Without a driving force (say stirring), the turbulence decays.
- $\mathbf{v}(\mathbf{r}_0, t)$ appears random in time and highly disordered in space.
- Turbulent flows exhibit a wide range of length scales: from the system size, size of obstacles, through large vortices down to the smallest ones at the Taylor microscale (where dissipation occurs).
- $\mathbf{v}(\mathbf{r}_0, t)$ are very different in distinct experiments with approximately the same ICs/BCs. But the time average $\bar{\mathbf{v}}(\mathbf{r}_0)$ is the same in all realizations.



- Unlike individual flow realizations, statistical properties of turbulent flow are reproducible and determined by ICs and BCs.
- As $\mathcal{R}$ is increased, symmetries (rotation/reflection/translation) are broken, but can be restored in a statistical sense in fully developed turbulence.

Lewis Richardson, Andrei Kolmogorov and Lars Onsager

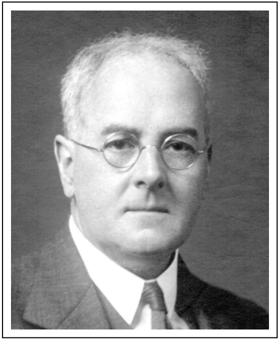
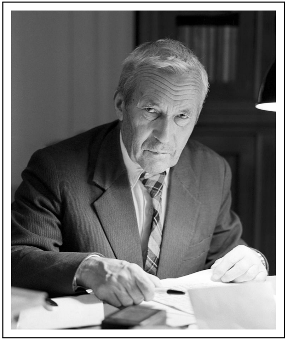
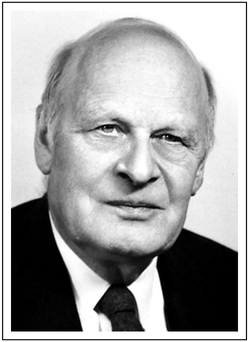




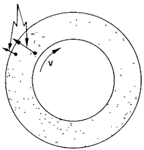
Figure: Lewis Richardson (L), Andrei Kolmogorov (C) and Lars Onsager (R).

Big whirls have little whirls that feed on their velocity,
and little whirls have lesser whirls and so on to viscosity.

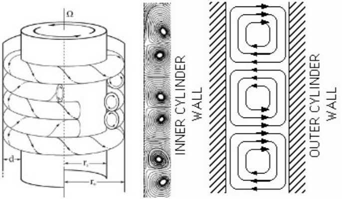
– L F Richardson, Weather Prediction by Numerical Process (1922).

Taylor experiment: flow between rotating cylinders

- Oil with Al powder between concentric cylinders $a \le r \le b$. Inner cylinder rotates slowly at ω_a with outer cylinder fixed. Oil flows steadily with azimuthal v_ϕ dropping radially outward from $\omega_a r_a$ to zero at $r = b$.
- Shear viscosity transmits v_ϕ from inner cylinder to successive layers of fluid. Centrifugal force tends to push inner layers outwards, but inward pressure due to wall and outer layers balance it. So pure azimuthal flow is stable.

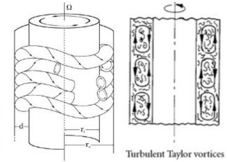


- When $\omega_a > \omega_{critical}$, flow is unstable to formation of toroidal Taylor vortices superimposed on the circumferetial flow. Translation invariance with z is lost. Fluid elements trace helical paths.
- Above $\omega_{critical}$, inward pressure and viscous forces can no longer keep centrifugal forces in check. The outer layer of oil prevents the whole inner layer from moving outward, so the flow breaks up into horizontal Taylor bands.

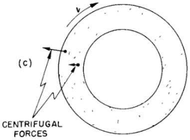


Taylor experiment: flow between rotating cylinders

- If ω_a is further increased, keeping $\omega_b = 0$ then # of bands increases, they become wavy and go round at $\approx \omega_a/3$. Rotational symmetry is further broken though flow remains laminar.



- At sufficiently high ω_a, flow becomes fully turbulent but time average flow displays approximate Taylor vortices and cells.
- There are 3 convenient dimensionless combinations in this problem: $(b-a)/a$, L/a and the Taylor number $Ta = \omega_a^2 a(b-a)^3/\nu^2$.
- For small annular gap and tall cylinders, Taylor number alone determines the onset of Taylor vortices at $Ta = 1.7 \times 10^4$.
- If the outer cylinder is rotated at ω_b holding inner cylinder fixed ($\omega_a = 0$), no Taylor vortices appear even for high ω_b. Pure azimuthal flow is stable.
- When outer layers rotate faster than inner ones, centrifugal forces build up a pressure gradient that maintains equilibrium.



Geoffrey Ingram Taylor

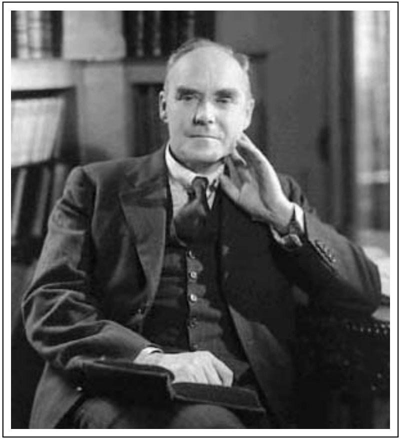
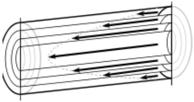


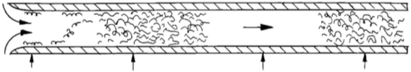
Figure: Geoffrey Ingram Taylor

Reynolds' expt (1883): Pipe flow transition to turbulence

- Consider flow in a pipe with a simple, straight inlet. Define the Reynolds number $\mathcal{R} = Ud/\nu$ where pipe diameter is d and U is flow speed.
- At very low $\mathcal{R}$ flow is laminar: steady Poiseuille flow (parabolic vel. profile).
- In general, turbulence in the pipe seems to originate in the boundary layer near the inlet or from imperfections in the inlet.



- If $\mathcal{R} \lesssim 2000$, any turbulent patches formed near the inlet decay.



- When $\mathcal{R} \gtrsim 10^4$ turbulence first begins to appear in the annular boundary layer near the inlet. Small chaotic patches develop and merge until turbulent 'slugs' are interspersed with laminar flow regions.
- For $2000 \lesssim \mathcal{R} \lesssim 10,000$, the boundary layer is stable to small perturbations. But finite amplitude perturbations in the boundary layer are unstable and tend to grow along the pipe to form fully turbulent flow.

Shocks in compressible flow

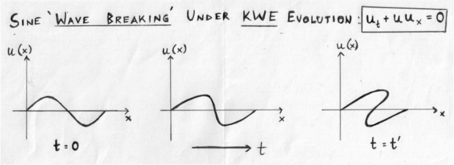
- A shock is usually a surface of small thickness across which $\mathbf{v}, p, \rho$ change significantly: modelled as a surface of discontinuity.
- Shock moves faster than the speed of sound. Roughly, if shock propagates sub-sonically, it could emit sound waves ahead of the shock that eliminate the discontinuity. Mach number $M = v_1/c > 1$.
- Sudden localized explosions like supernovae or bombs often produce spherical shocks called blast waves. Nature of spherical blast wave from atom bomb was worked out by Sedov and Taylor in the 1940s.
- Material from undisturbed medium in front of shock (ρ_1) moves behind the shock and gets compressed to ρ_2.
- Fluxes of mass, momentum and energy are equal in front of and behind the shock. This may be used to relate ρ_1, v_1, p_1 to ρ_2, v_2, p_2. These lead to the Rankine-Hugoniot 'jump' conditions.
- Viscous term $\nu \nabla^2 \mathbf{v}$ is often important in a shock since $\mathbf{v}$ changes rapidly. Leads to heating of the gas and entropy production.

1d toy models: KWE and Burgers' regularization

- 1d Kinematic Wave Equation (KWE) models shocks and traffic flow

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{24}$$

- KWE is a time-reversible, non-linear advection equation for inviscid flow.
- It has ∞ of conserved quantities: momentum $\int u\, dx$, energy $\int u^2\, dx \ldots$
- But $u(x,t)$ can develop large gradients, u_x may diverge and u may even become multi-valued.



- Burgers modified KWE by adding a viscous term $u_t + uu_x = \nu u_{xx}$.
- When u becomes steep, νu_{xx} becomes significant and prevents shock-like singularities. Energy is lost to viscous dissipation.
- $\int u\, dx =$ const. but energy monotonically decays $\frac{d}{dt}\int \frac{1}{2}u^2\, dx = -\nu \int u_x^2\, dx$
- Burgers' dissipative regularization is not time-reversible unlike KWE.

Conservative regularization of KWE: KdV equation

- KdV models nonlinear non-dissipative dispersive water waves

$$u_t - 6uu_x + u_{xxx}, \quad u = \text{height of wave}. \tag{25}$$

- KdV admits infinitely many conservation laws, a Hamiltonian and Poisson bracket formulation and is exactly solvable via IST.

$$\dot{u} = \{u, H\}, \quad H = \int \left[\frac{1}{2} u_x^2 + u^3 \right] dx, \quad \{u(x), u(y)\} = \frac{1}{2}(\partial_x - \partial_y)\delta(x-y). \tag{26}$$

- It also has a discrete 'time-reversal' (PT) symmetry $t \to -t, x \to -x, u \to u$.
- The 3rd order dispersive term prevents development of large gradients of u. Smooth solutions exist for arbitrarily long times. Displays recurrent motions in bounded domains.
- KdV has cnoidal (periodic) and finitely many interacting, exact N-soliton solutions: competing effects of non-linear advection and dispersion result in soliton preserving its shape under time evolution.

Diederik Johannes Korteweg and Gustav de Vries

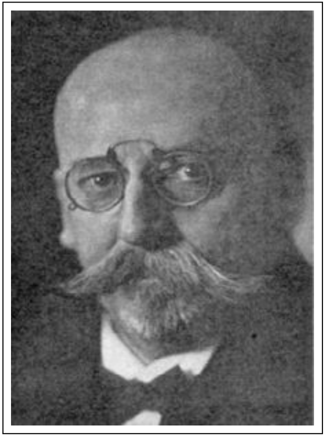


Figure: Diederik Johannes Korteweg (left) and Gustav de Vries (right).

Conservative regularization of compressible 3d Euler

- Ideal Eulerian evolution tends to stretch vortex tubes leading to vortical singularities. Viscous dissipation can regularize unbounded growth of $\mathbf{w}$.
- Is there a KdV-like conservative regularization in 3d? With A Thyagaraja and S Sachdev, we have found a minimal local dispersive regularization!

$$\rho(\partial_t + \mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p - \lambda^2 \rho \mathbf{w} \times (\nabla \times \mathbf{w}) \quad \text{and} \quad \rho_t + \nabla\cdot(\rho \mathbf{v}) = 0. \tag{27}$$

- The twirl force is the simplest conservative regularization: lowest number of derivatives and non-linearity. 2nd order in $\mathbf{v}$, just like NS.
- It kicks in when $\mathbf{w}$ has large gradients and prevents unbounded growth of enstrophy $\int \mathbf{w}^2 \, d\mathbf{r}$ as swirl energy E^* is conserved

$$E^* = \int \left[\frac{1}{2}\rho \mathbf{v}^2 + U(\rho) + \frac{1}{2}\lambda^2 \rho \mathbf{w}^2 \right] d\mathbf{r} \quad \text{where} \quad U = \frac{p}{\gamma - 1} \quad \text{with} \quad \lambda^2 \rho = \text{constant}. \tag{28}$$

- Short distance regulator λ is like a position-dependent mean free path: smaller in denser regions. Corresponding regulator in NS is viscosity ν.
- Twirl force $-\lambda^2 \rho \mathbf{w} \times (\nabla \times \mathbf{w})$ is the vortical analogue of the magnetic Lorentz force term $\mathbf{j} \times \mathbf{B} = -\frac{1}{\mu_0} \mathbf{B} \times (\nabla \times \mathbf{B})$ arising in MHD with $\lambda^2 \rho \leftrightarrow \frac{1}{\mu_0}$.

Conservative regularization of compressible 3d Euler

- Twirl regularization preserves parity, $t \to -t$ and Galilean symmetries.
- Local conservation laws for mass, energy, $\mathbf{P}$, $\mathbf{L}$ and helicity $\mathbf{w} \cdot \mathbf{v}$.
- The vorticity is frozen into a 'swirl' velocity field $\mathbf{v}_* = \mathbf{v} + \lambda^2 \nabla \times \mathbf{w}$ leading swirl-Kelvin theorem for surfaces S_t^* and contours C_t^* moving with $\mathbf{v}_*$

$$\frac{d}{dt} \int_{S_t^*} \mathbf{w} \cdot dS = 0 \qquad \text{or} \qquad \frac{d}{dt} \oint_{C_t^*} \mathbf{v} \cdot d\mathbf{l} = \frac{d\Gamma}{dt} = 0 \quad \text{where} \quad C_t^* = \partial S_t^*.$$

- R-Euler follows from Landau-Morrison-Greene PBs with $H = E^*$:

$$\{\rho(x), v(y)\} = -\nabla_{\mathbf{x}} \delta(x-y), \quad \{v_i(x), v_j(y)\} = \epsilon_{ijk} w_k \delta(x-y)/\rho, \quad \{\rho(x), \rho(y)\} = 0.$$

- Also discovered new regularization terms allowing us to bound higher moments of $\mathbf{w}$ and $\nabla \times \mathbf{w}$.
- Steady R-Euler used to model rotating vortex, twirl term smooths out discontinuity in $\mathbf{w}$ at edge of tornado on length-scales of $\mathcal{O}(\lambda)$ like $\hbar$ regularizes classical singularities. Radial drop in $\mathbf{w}$ is related to an increase in ρ through regulator λ.
- Extended regularization to compressible magnetohydrodynamics to model charged fluids (plasmas).

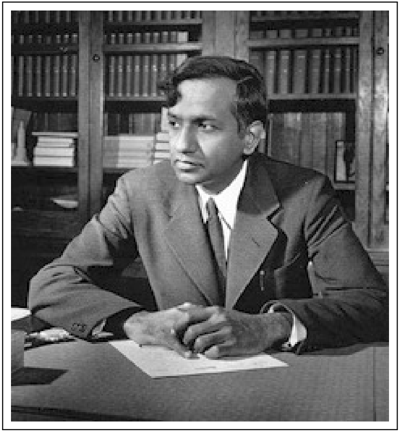
Added-mass Higgs mechanism analogy

- With S Phatak, we have developed a new physical correspondence between the Higgs mechanism and the added mass effect.
- Consider translational motion a rigid body of mass m in an inviscid, incompressible and irrotational fluid at rest in 3d space. To impart acceleration $\dot{U}$, external agent must apply force $F_i = ma_i + \mu_{ij}a_j$
- Added mass force $\mu_{ij}a_j$ is proportional to acceleration but could point in a different direction, depending on shape of body.
- The constant 3×3 symmetric matrix is the added mass tensor. It depends on fluid density and shape of body.
- μ_{ij} is isotropic for a sphere and equal to half mass of displaced fluid. Added mass roughly grows with cross-sectional area presented by body. Flat plate has no added mass when accelerated along its plane.
- In the Higgs mechanism, the otherwise massless gauge vector bosons (W, Z) acquire masses by interacting with a scalar field. W, Z, photon correspond to directions in the Lie algebra of the gauge group.

Added Mass Effect –vs– Higgs Mechanism

rigid body	gauge bosons
fluid	scalar (Higgs) field
space occupied by fluid	Lie algebra of gauge group G
dimension of container	dimension of Lie algebra $\underline{G}$
directions of $\vec{\mathbf{a}}$ relative to body	various directions in Lie algebra
number of flat directions	dimension of unbroken subgrp H
added mass tensor μ_{ij}	mass matrix M_{ab}
motion along flat face	massless photon
added mass eigenvalues	masses of vector bosons
sphere moving in 3d	SU$(2) \to \{1\}$, scalar doublet
hollow cylinder in 3d	SO$(3) \to$ SO(2), scalar triplet
broken pressure symmetry	broken gauge symmetry
fluid density ρ $[\mu_{ij}^{\text{sphere}} \propto \rho R^3 \delta_{ij}]$	scalar vev $[M_{ab}^{\text{SU}(2)} \propto \eta^2 g^2 \delta_{ab}]$
$F_i - m a_i \; = \; \mu_{ij}\, a_j$	$-j^\nu + \partial_\mu F^{\mu\nu} = g^2 \langle\phi\rangle^2 A^\nu$
density fluctuations, Mach expan.	quantum fluctuations, loop expan.
accelerating body ‘carries’ a flow	W boson carries Goldstone mode.
compressional wave around body	Higgs particle

Prominent Indian fluid dynamicists



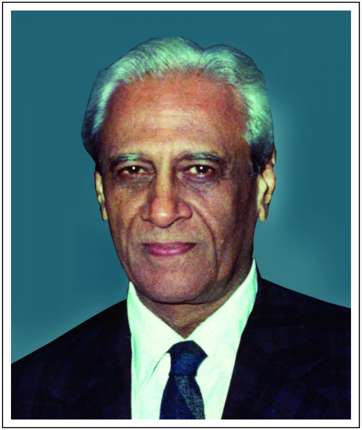
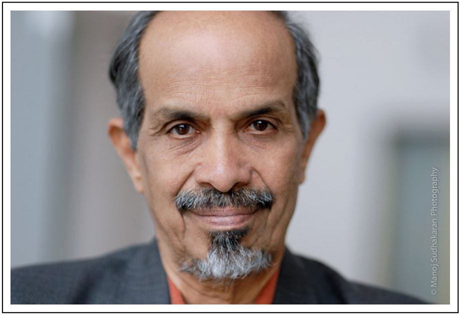


Figure: Subrahmanyan Chandrasekhar (left) and Satish Dhawan (right).

Prominent Indian fluid dynamicists



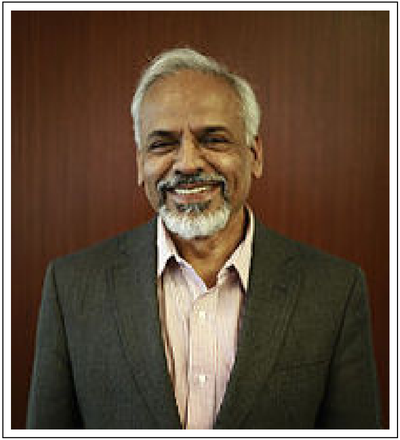
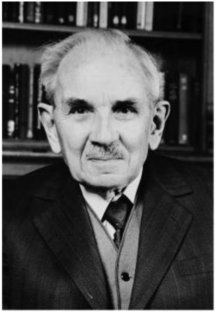


Figure: Roddam Narasimha (left) and Katepalli Sreenivasan (right).

Existence & Regularity: Clay Millenium Problem

- Either prove the existence and regularity of solutions to incompressible NS subject to smooth initial data [in $\mathbb{R}^3$ or in a cube with periodic BCs] OR show that a smooth solution could cease to exist after a finite time.
- J Leray (1934) proved that weak solutions to NS exist, but need not be unique and could not rule out singularities.
- Hausdorff dim of set of space-time points where singularities can occur in NS cannot exceed one. So hypothetical singularities are rare!
- O Ladyzhenskaya (1969) showed existence and regularity of classical solutions to NS regularized with hyperviscosity $-\mu(-\nabla^2)^\alpha \mathbf{v}$ with $\alpha \geq 2$. J-L Lions (1969) extended it to $\alpha \geq 5/4$.
- A proof of existence/uniqueness/smoothness of solutions to NS or a demonstration of finite time blow-up is mathematically important.
- Physically, it is know that for large enough $\mathcal{R}$, most laminar flows are unstable, they become turbulent and seem irregular. Methods to calculate/predict features of turbulent flows would also be very valuable.

Jean Leray, Olga Ladyzhenskaya and Jacques Louis Lions



Jean Leray - Royal society (1991)

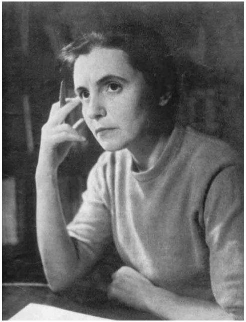


Figure: Jean Leray (left), Olga Ladyzhenskaya (middle) and Jacques Louis Lions (right).

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von Karman vortex street in the clouds

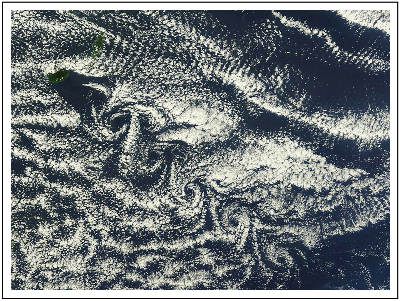

Figure: von Karman vortex street in the clouds above Yakushima Island

Thank you!