Mathematical Methods, Spring 2025 CMI

Assignment 9

Due by the beginning of the class on Wednesday Apr 23, 2025

Induced metric, Laplace-Beltrami operator, divergence

1. $\langle \mathbf{3+3+2} \rangle$ Consider the embedding $f : T^2 \to \mathbb{R}^3$ of the 2-torus in $\mathbb{R}^3$ (with Cartesian coordinates) given by $(\theta, \varphi) \mapsto ((R + r\cos\theta)\cos\varphi, (R + r\cos\theta)\sin\varphi, r\sin\theta)$ where $R > r > 0$ are fixed. (a) Indicate the meanings of r, R, θ, φ by drawing a figure. (b) Find the pullback f^*g of the Euclidean metric $g_{ij} = \delta_{ij}$ to obtain an induced metric on the torus. (c) Find the corresponding Riemannian volume form on T^2 expressed in the θ, ϕ coordinates.

2. $\langle \mathbf{1+2} \rangle$ Consider plane polar coordinates θ, ϕ on the Euclidean plane with the negative horizontal axis and origin removed: $x = r\cos\theta$ and $y = r\sin\theta$. (a) Find the components of the metric $g = dx \otimes dx + dy \otimes dy$ in polar coordinates. (b) Find a formula for the Laplace-Beltrami operator Δf acting on the scalar field $f(r, \theta)$.

3. $\langle \mathbf{2+1+3+1} \rangle$ Consider the metric $g = (dx \otimes dx + dy \otimes dy)/y^2$ on the upper half plane $U : (x, y) \in \mathbb{R}^2$ with $y > 0$. (a) Suppose $v = v^x \partial_x + v^y \partial_y$ is a vector field on U. Find a formula for its divergence $\operatorname{div} v$. (b) Find the Riemannian volume form ω corresponding to this metric. (c) The Lie derivative of a 2-form along a vector field v is defined as the 2-form with components

$$(\mathcal{L}_v \omega)_{jk} = v^i \partial_i \omega_{jk} + (\partial_j v^i)\omega_{ik} + (\partial_k v^i)\omega_{ji}. \tag{1}$$

   Find $\mathcal{L}_v \omega$ for the Riemannian volume form ω obtained earlier. Hint: How many independent components does $\mathcal{L}_v \omega$ have? (d) Since $\mathcal{L}_v \omega$ is a top order form on a 2d manifold, it must be a multiple of the volume form ω by some smooth function f depending on v. Find f and interpret it in this example.