**Mathematical Methods, Spring 2025 CMI**
Assignment 8
Due by the beginning of the class on Friday Apr 4, 2025
Differential forms

1. $\langle \mathbf{2+2} \rangle$ **Closed, exact forms.** Consider the 1-form $\alpha = (x dx + y dy)/r^2$ on the punctured $x$-$y$ plane $\mathbb{R}^2 \setminus (0,0)$ where $r^2 = x^2 + y^2$. (a) Show that the exterior derivative $d\alpha = 0$. Thus, $\alpha$ is closed. (b) Find a smooth function $f(x,y)$ on the punctured plane such that $\alpha = df$, thereby establishing that $\alpha$ is exact.

2. $\langle \mathbf{2+2+4} \rangle$ **Integrating denominator.** Consider the 1-form on $\mathbb{R}^2$, $\omega(x,y) = y\,dx - x\,dy$. Since $d(df) = d^2 f = 0$, a necessary 'integrability' condition for $\omega$ to be exact ($\omega = d\sigma$ for some $\sigma$) is that it be closed ($d\omega = 0$). (a) Show that $\omega$ disobeys the integrability condition. (b) Look for an integrating denominator $\tau(x,y)$ and function $\sigma(x,y)$ such that $\omega/\tau = d\sigma$ is exact on some open subset of $\mathbb{R}^2$. (c) Examine whether $\tau$ is unique. Try to find the form of all such integrating denominators $\tau$. Hint: Try to derive a differential equation for $\tau$. Remark: For a class of 1-forms $\omega$ associated to the infinitesimal heat added reversibly to a system, the $2^{\text{nd}}$ law of thermodynamics guarantees the existence of an integrating denominator $\tau$, which is the absolute temperature. $\sigma$ is the entropy function on the thermodynamic state space.

3. $\langle \mathbf{1+2+2} \rangle$ **Volume form.** Consider $\mathbb{R}^2$ with Cartesian coordinates $x^i = (x,y)$ with the 'volume' (area) form $\Omega = \frac{1}{2}\Omega_{ij} dx^i \wedge dx^j$ where $\Omega_{ij} = \epsilon_{ij}$, the Levi-Civita symbol. (a) Show that $\Omega = dx \wedge dy$. (b) Suppose we change to polar coordinates $\tilde{x}^i = (r,\theta)$ via $x = r\cos\theta, y = r\sin\theta$ on the overlap of the coordinate systems. Find all the components $\tilde{\Omega}_{ij}$ in the polar system. (c) Express $\Omega = \rho(r,\theta) dr \wedge d\theta$ for suitable volume density $\rho$ and relate $\rho$ to the Jacobian.