Mathematical Methods, Spring 2025 CMI
Assignment 7
Due by the beginning of the class on Friday Mar 14, 2025
manifolds, vector fields

1. $\langle \mathbf{2+2+2+2+2} \rangle$ Consider the unit sphere S^2 embedded in $\mathbb{R}^3$: $x^2+y^2+z^2=1$. Let $N=(0,0,1)$ and $S=(0,0,-1)$ be the North and South poles and E the equatorial plane $z=0$. From Assignment 1, we are familiar with the 'stereographic projection' from N and S to $E \cong \mathbb{R}^2$. Let (X,Y) be the stereographic coordinates of $(x,y,z) \neq N$ on S^2 obtained via the stereographic projection from N and let (X',Y') be the coordinates of $(x,y,z) \neq S$ via the projection from S. (a) What are the associated coordinate neighborhoods on S^2 and where do they overlap? (b) Is the overlap path connected? Is the overlap simply connected? (c) Give expressions for (X,Y) and (X',Y') in terms of x,y,z (you may use earlier results). (d) Find the coordinate transformation (transition functions) expressing (X,Y) in terms of (X',Y') and vice versa. (e) Briefly indicate why we may regard S^2 as a smooth manifold.

2. $\langle \mathbf{4} \rangle$ Propose an example of a nontrivial (not the identity) smooth diffeomorphism of the circle $g : S^1 \to S^1$. You do not need to specify it in an atlas of coordinate charts if you unambiguously specify the map g and its inverse and indicate why it is a diffeomorphism.

3. $\langle \mathbf{2+2} \rangle$ Here, we work within a coordinate patch on an n-dimensional manifold M with coordinates x^i. (a) Show that the Lie derivative satisfies the Leibniz rule: $\mathcal{L}_u(fv) = (\mathcal{L}_u f)v + f\mathcal{L}_u v$ for a smooth function f and vector fields u,v. (b) What is $\mathcal{L}_{fu}v - f\mathcal{L}_u v$?

4. $\langle \mathbf{4} \rangle$ Use the formula for the commutator of vector fields in a patch on an n-dimensional manifold M with coordinates $x = (x^1, \cdots, x^n)$ to show that they satisfy the Jacobi identity $[u,[v,w]] + [w,[u,v]] + [v,[w,u]] = 0$.

5. $\langle \mathbf{3+3} \rangle$ Consider the vector field on the Euclidean plane $\mathbb{R}^2$ with Cartesian coordinates (x,y) given by $v = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$. (a) Sketch the vector field on the x-y plane by displaying arrows. Which way does it point at the origin? For illustration purposes, you may make the lengths of the arrows proportional to the Euclidean lengths of the tangent vectors (with ∂_x and ∂_y regraded as having unit length). (b) Solve for the integral curve $(x(t),y(t))$ through the point $(x(0)=1, y(0)=0)$ and plot it. Describe all the integral curves: what curves are they and which way do they point?