Mathematical Methods, Spring 2025 CMI
Assignment 6
Due by the beginning of the class on Friday Feb 28, 2025
Entire functions, Gamma function

1. $\langle \mathbf{2+2+2+2+2+2+2+2} \rangle$ An entire function $f(z)$ must admit a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with infinite radius of convergence. For this, the coefficients must decay sufficiently fast. Consider the following types of coefficient sequences (for $n \geq 1$). (i) $a_n = 1/n^k$ for $k > 0$, (ii) $a_n = e^{-\alpha n}$ for $\alpha > 0$ and (iii) $a_n = n^{-n}$ (regarded as the asymptotic behavior of $1/n!$) (iv) $a_n = n^{-\beta n}$ for $\beta > 0$ (regarded as the asymptotic behavior of $(n!)^{-\beta}$). (a) Give a descriptive name for each type of decaying sequence. (b) In each case, find the radius R of convergence. (c) The (coefficient) order of an entire function is defined as
$$\rho = \limsup_{n\to\infty} \frac{n \log n}{\log(1/|a_n|)}. \tag{1}$$
Find the orders of the entire functions among the ones in (i)–(iv). (d) What is the order of a polynomial, regarded as an entire function? (e) What are the orders of the entire functions e^z and $\sin z$? (f) Give an example of an entire function of order two (closed form expression). (g) Propose a coefficient sequence which represents an entire function of order $1/2$. (h) Give a simple example (closed form expression) of an entire function of order half.

2. $\langle \mathbf{1+2+2+4+1} \rangle$ We have shown that the Gamma function has simple poles at $z = 0, -1, -2, \ldots$ and nowhere else. It is also possible to show that $\Gamma(z)$ does not vanish anywhere (you may assume this). Thus, $1/\Gamma(z)$ is an entire function (this is also a consequence of the contour integral representation of $1/z!$ developed in Assignment 5.). Thus, $1/\Gamma(z)$ is an entire function whose only zeros are simple ones at $z = 0, -1, -2, -3, \cdots$. (a) Suppose we are given the Hadamard canonical infinite product for the reciprocal Gamma function:
$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{k=1}^{\infty} (1 + z/k) e^{-z/k} \quad \text{where } \gamma \text{ is Euler's constant defined below.} \tag{2}$$
Use it to identify the genus of $1/\Gamma(z)$. (b) Where are the zeros of the entire function $1/\Gamma(1-z)$ and what do you think its genus must be? (c) Guess/conjecture a possible relation between $1/[\Gamma(z)\Gamma(1-z)]$ and a familiar entire trigonometric function, based on the zeros and genus. (d) Write down an infinite product P for $1/\Gamma(1-z)$ by substituting in (2). By viewing the infinite product P as the limit of partial products ($P_N = \prod_{k=1}^{N} \cdots$), argue that
$$\frac{1}{\Gamma(1-z)} = e^{-\gamma z} \prod_{k=1}^{\infty} (1 - z/k) e^{z/k}. \tag{3}$$
Hint: Use $\lim_{N\to\infty}(1 + \frac{1}{2} + \cdots + \frac{1}{N} - \log N) = \gamma$. (e) Use these infinite products to check/correct the conjectured formula in (c).