Mathematical Methods, Spring 2025 CMI
Assignment 5
Due by the beginning of the class on Wednesday Feb 19, 2025
Hilbert transform, branch cut

1. $\langle \mathbf{2{+}4} \rangle$ Consider the kernel of the square of the Hilbert transform $\epsilon^2(x,y) = \mathcal{P}\int_{-\infty}^{\infty} \epsilon(x,z)\epsilon(z,y)dz$ where $\epsilon(x,z) = \frac{1}{\pi(x-z)}$. (a) Show that $\epsilon^2(x,y)$ is a symmetric function of its arguments. (b) Evaluate this Cauchy principal value using complex contour integration and show that it vanishes for $x \neq y$ (both fixed real numbers).

2. $\langle \mathbf{2{+}2{+}2{+}2{+}2{+}3} \rangle$ (a) Consider the function $f(w) = e^w/w^{n+1}$ for $n = 0, 1, 2, \cdots$. Find the poles of f, their orders and residues. (b) Deduce the contour integral representation

$$\frac{1}{n!} = \frac{1}{2\pi i}\oint_C \frac{e^w}{w^{n+1}}dw \quad \text{for} \quad n = 0, 1, 2, \cdots. \tag{1}$$

Specify a suitable contour C. (c) Use this to extend the definition of $1/n!$ to negative integers and evaluate it for $n = -1, -2, -3, \cdots$. (d) Now we extend n to a complex variable, denoted z. Identify the branch points and branch cut of w^{z+1} in the complex w plane for fixed noninteger z by relating it to familiar multivalued functions we have discussed. (e) Find the values of the principal branch of w^{z+1} just above and below the branch cut. (f) Restrict to the principal branch and propose a contour integral definition of $1/z!$ for more general complex z. Specify a new contour C and a suitable generalization of (1).