Mathematical Methods, Spring 2025 CMI

Assignment 4

Due by the beginning of the class on Monday Feb 10, 2025

residue calculus, integral kernel, Fourier transform

1. $\langle \mathbf{7} \rangle$ Use residue calculus to find the sum of the series $A = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$. Hint: Find the residues of $\pi \csc \pi z$. Note: It is possible to manipulate the sum $S = \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$ to find A, but do not do so except perhaps to check your answer.

2. $\langle \mathbf{1+2+2} \rangle$ Find the integral kernels $A(x,y), B(x,y), C(x,y)$ of the following differential operators that act on smooth real-valued functions $f(x)$ that vanish sufficiently fast as $x \to \pm\infty$ on the real line. (a) $(Af) = \kappa(x) f(x)$ for some function $\kappa(x)$, (b) $(Bf)(x) = f'(x)$ and (c) $(Cf)(x) = f''(x)$.

3. $\langle \mathbf{2+1+1+1+3} \rangle$ (a) Find the (inverse) Fourier transform $\psi_+(x) = \int_{-\infty}^{\infty} \tilde{\psi}_+(p) e^{ipx} dp/2\pi$ of the momentum space wave function $\tilde{\psi}_+(p) = \theta(p \geq 0)\, e^{-p}$ where θ is the Heaviside step function (one for $p \geq 0$, zero otherwise). (b) Discuss the domain of analyticity and singularities of $\psi_+(x)$ viewed as a function of a complex position variable x. (c) Suppose $\tilde{\psi}_-(p) = e^p\, \theta(p \leq 0)$. Find its inverse Fourier transform $\psi_-(x)$. (d) To which half of the complex x-plane can $\psi_-(x)$ be extended as an analytic function free of singularities? (e) Suppose $\tilde{\psi}(p)$ vanishes on the real momentum axis outside a finite interval $[a, b]$. Guess (with brief reasoning) the singularity structure of its inverse Fourier transform $\psi(x)$, viewed as an analytic function of the complex position x.