Mathematical Methods, Spring 2025 CMI

Assignment 3

Due by the beginning of the class on Feb 3, 2025

poles, holomorphy, harmonic functions

1. $\langle \mathbf{2+3+1} \rangle$ (a) Show that the zeros of $\sin z$ must be real. (b) Find the poles (mentioning the order of each pole) of $\cot z$ and the residues at the poles. (c) Find the poles and residues of $f(z) = \pi \cot \pi z$.

2. $\langle \mathbf{4+1} \rangle$ Suppose we are told that $\partial_z f(x,y)$ is a holomorphic function in some domain D. (a) Try to figure out what kind of function $f(x,y)$ can be in general. Propose an expression (no need to try to show whether this is the only possibility). (b) Give a simple example of a function $g(x,y)$ that is not of the proposed form and for which $\partial_z g$ is not holomorphic.

3. $\langle \mathbf{2+2+4} \rangle$ (a) Find two different factorizations of the Laplace operator $\partial_x^2 + \partial_y^2$ as a product (composition) of two differential operators. (b) Use these to propose (conjecture) an additive decomposition of (complex-valued) harmonic functions. [A complex-valued function $f(x,y)$ is harmonic if $\partial_x^2 f + \partial_y^2 f = 0$.] (c) Briefly draw an analogy between the above factorization and decomposition with those for the one dimensional wave operator $\frac{1}{c^2}\partial_t^2 - \partial_x^2$ and d'Alembert's solution of the wave equation $\frac{1}{c^2}\psi_{tt} = \psi_{xx}$ for x on the real line. Here, $\psi(x,t)$ is a C^2 real function and $c > 0$ is a constant speed.