Mathematical Methods, Spring 2025 CMI
Assignment 2
Due by the beginning of the class on Jan 22, 2025
sequences, power series

1. $\langle \mathbf{3} \rangle$ Use power series to show that $e^{z+w} = e^z e^w$ for a pair of complex variables z and w.

2. $\langle \mathbf{1+1+1} \rangle$ Consider the sequence for $n = 0, 1, 2, \ldots$ defined as

$$a_n = \begin{cases} 1 & \text{if} \quad n = 2^k \quad \text{for some} \quad k = 0, 1, 2, 3, \ldots \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

   (a) List out the first ten a_n. (b) Find the limit points of this sequence. (c) Find $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$.

3. $\langle \mathbf{1+1+1+1+1+2+2} \rangle$ Consider the power series $f(z) = z + z^2 + z^4 + z^8 + \cdots + z^{2^k} + \cdots$. (a) Explain why this series is absolutely convergent. (b) Find the radius of convergence R by an application of the formula $1/R = \limsup_{n\to\infty} |a_n|^{1/n}$ (c) Find the smallest positive real value of z for which the series diverges, we will call it the first singularity of f. (d) Find a functional equation for f. Hint: look for a relation between $f(z)$ and $f(z^2)$ (e) Find another singular point of f using the functional equation. (f) Repeat to find another functional equation and two more singular points. (g) Argue that f has singularities at all $(2^k)^{\text{th}}$ roots of unity. Classify these as isolated/nonisolated singularities.