Mathematical Methods, Spring 2025 CMI

Assignment 10

Due by 4pm on Friday May 2, 2025

Lie group, Lie algebra

1. $\langle \mathbf{2+2+2+3} \rangle$ Characterize the matrices that lie in the Lie algebras of the following matrix groups and use this to find the real dimensions of the corresponding Lie groups: (a) the real general linear group $GL_n(\mathbb{R})$, (b) the special linear group $SL_n(\mathbb{R})$ which is the subgroup with unit determinant (c) the orthogonal group $O(n)$, (d) $SU(n)$.

2. $\langle \mathbf{4} \rangle$ Suppose a Lie algebra has a basis of 'generators' $e_1, \cdots, e_n$ that satisfy the Lie brackets $[e_i, e_j] = \sum_{k=1}^n c_{ij}^k e_k$. Find the condition on the structure constants c_{ij}^k that encodes the Jacobi identity.

3. $\langle \mathbf{3+1+2} \rangle$ Consider the 'adjoint representation' of a matrix Lie group G on its Lie algebra $\underline{G}$, given by $Ad_g(v) = gvg^{-1}$ for $g \in G$ and $v \in \underline{G}$. (a) Show that the adjoint representation is a group action and that the action is linear on $\underline{G}$. (b) Suppose $g \approx I + u$ is infinitesimally close to the group identity. Find an approximate formula for g^{-1}. (c) Use the group adjoint action to deduce an action of the Lie algebra $\underline{G}$ on itself. Give a formula for this action.

4. $\langle \mathbf{2+3+4} \rangle$ Consider the following invertible linear map between the cross product Lie algebra on $\mathbb{R}^3$ and the SU(2) Lie algebra. If $\boldsymbol{x} \in \mathbb{R}^3$ is a vector, then the corresponding element of $\underline{SU(2)}$ is the traceless antihermitian matrix $u(\boldsymbol{x}) = \boldsymbol{\tau} \cdot \boldsymbol{x}$. Here $\tau_j = \sigma_j / 2i$ for $j = 1, 2, 3$ and σ_j are the three Pauli matrices. (a) Find the Lie brackets among the SU(2) Lie algebra basis elements: $[\tau_i, \tau_j]$. (b) Find the inverse map from $\underline{SU(2)} \to \mathbb{R}^3$. Give an explicit formula for the components of $\boldsymbol{x}(u)$. (c) Show that this map is a homomorphism of Lie algebras: roughly, it takes the cross product to the commutator. Give the precise formula expressing this condition, say what it means in words and show that it holds. Hint. You may use the identity $\sigma_j \sigma_k = i \epsilon_{jkl} \sigma_l + \delta_{jk} I_{2\times 2}$ where the repeated index is summed from 1 to 3 and i is the imaginary unit.