Mathematical Methods, Spring 2025 CMI

Assignment 1

Due by the beginning of the class on Jan 15, 2025

Stereographic projection

1. $\langle \mathbf{3+3+3} \rangle$ Consider the unit sphere S^2 embedded in $\mathbb{R}^3$: $x_1^2 + x_2^2 + x_3^2 = 1$. Let $N = (0,0,1)$ be the North pole of the sphere and consider the equatorial plane (thought of as the complex plane) $\mathbb{C}$ defined by $x_3 = 0$. Given any point $P = (x_1, x_2, x_3) \neq N$ on S^2, we define its **stereographic projection** to be the unique point $P' = (x, y) \in \mathbb{C}$ through which the line joining N and P passes (see Fig. 1). (a) Express the coordinates (x, y) of the stereographic projection of P in terms of x_1, x_2, x_3. Hint: use the fact that N, P, P' are collinear. (b) Find the inverse map, expressing the coordinates of a point (x_1, x_2, x_3) on S^2 in terms of x and y. (c) Check that the formulae in (a) and (b) behave as expected when the point on the sphere approaches the North pole.

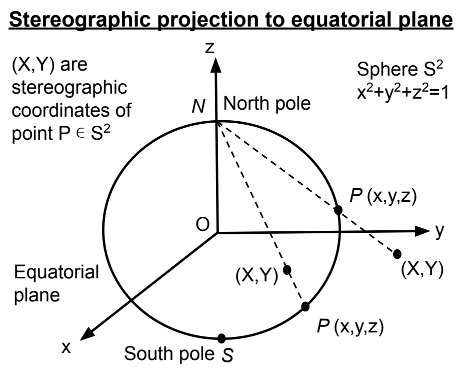

Figure 1: Stereographic projection. Here, (x, y, z) is to be read as (x_1, x_2, x_3) while (X, Y) is to be read as (x, y).

2. $\langle \mathbf{2+3+3} \rangle$ Suppose we define a (non-Euclidean) distance function on the extended complex plane $\hat{\mathbb{C}}$ by

$$d(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}} \tag{1}$$

This distance function is based on the length of the chord joining the corresponding points on the Riemann sphere. (a) Find an expression for the distance between a point $z \in \mathbb{C}$ and the point at infinity $z = \infty$. (b) Sketch the region in the complex plane consisting of points within a distance $R > 0$ from the point $z = \infty$. Mention any restriction on R and why it is reasonable, based on the information given. (c) Suppose z and $z + dz$ are infinitesimally separated, with $z = x + iy$. Find the square of the distance (denoted ds^2) between z and $z + dz$ by keeping only terms up to quadratic order in infinitesimals. Express the answer in terms of dx and dy as well as in terms of dz and $d\bar{z}$.