Mathematical Methods, Spring 2024 CMI

Assignment 9

Due by the beginning of the class (1030 am) on Tue, Mar 12, 2024

Geodesic equation on upper half plane

1. $\langle \mathbf{5+3+3+3+4+4} \rangle$ The upper half plane $U = \{(x,y) \in \mathbb{R}^2 | y > 0\}$ has the Poincaré metric $g = (dx \otimes dx + dy \otimes dy)/y^2$. (a) Find all the Christoffel symbols: Γ^x_{xx} etc. (b) Show that the resulting geodesic equations are the pair of ODEs (dots denote derivatives with respect to an affine parameter t)

$$\ddot{x} - \frac{2}{y}\dot{x}\dot{y} = 0 \quad \text{and} \quad \ddot{y} + \frac{1}{y}\dot{x}\dot{x} - \frac{1}{y}\dot{y}\dot{y} = 0. \tag{1}$$

(c) Establish that

$$p = \frac{\dot{x}}{y^2} \quad \text{and} \quad H = \frac{1}{2y^2}(\dot{x}^2 + \dot{y}^2) \tag{2}$$

are conserved along geodesics. (d) Eliminate x and derive a self-contained ODE for y. (e) Solve for the geodesics when $p = 0$. Describe their shape in Euclidean language. How long (in t) does it take the geodesics to approach the x axis? (f) When p is not necessarily zero, show that the geodesic equations may be brought to a Newtonian form $\ddot{u} = -V'(u)$ where $y = e^u$. What is a suitable potential $V(u)$?