Mathematical Methods, Spring 2024 CMI

Assignment 7

Due by the beginning of the class (1030 am) on Tue, Feb 27, 2024

Differential forms, induced metric

1. ⟨**4**⟩ Suppose A is an $N \times N$ **matrix-valued 1-form** on $\mathbb{R}^2$ with coordinates x^0 and x^1. What this means is that when written as a linear combination of coordinate basis 1-forms dx^μ, the coefficients A_μ are $N \times N$ matrices. Viewed differently, A is a matrix, each of whose entries is a 1-form. In components, $A^a_b = (A_\mu)^a_b dx^\mu$ where $\mu = 0, 1$ and $a, b = 1, \ldots, N$ are matrix indices. We use up-down matrix indices since a matrix may be viewed as a (1,1) tensor. Now, we may construct an $N \times N$ matrix-valued 2-form via the wedge product $A \wedge A$. Show that $A \wedge A$ is given by

$$A \wedge A = [A_0, A_1] dx^0 \wedge dx^1, \quad \text{where} \quad [A_0, A_1] = A_0 A_1 - A_1 A_0. \tag{1}$$

2. ⟨**3**⟩ Taking pressure p and volume V as independent variables for the internal energy U of a gas, use the 1^{st} law of thermodynamics to obtain an expression for the **heat 1-form** ω [associated to the infinitesimal heat added reversibly to a fixed mass of a (not necessarily ideal) gas].

3. ⟨**5+5**⟩ Consider the map (embedding) from the **2-torus** to 3d Euclidean space (with Cartesian coordinates) $f : T^2 \to \mathbb{R}^3$ given by $(\theta, \varphi) \mapsto ((R + r\cos\theta)\cos\varphi, (R + r\cos\theta)\sin\varphi, r\sin\theta)$ where $R > r > 0$ are fixed. (a) Indicate the meanings of r, R, θ, φ by drawing a figure of the torus. (b) Find the pullback f^*g of the Euclidean metric $g_{ij} = \delta_{ij}$, i.e., find the induced metric on the torus.