Mathematical Methods, Spring 2024 CMI

Assignment 6

Due by the beginning of the class (1030 am) on Tue, Feb 20, 2024

Volume form, pushforward, pullback

1. $\langle \mathbf{3+4+2} \rangle$ **Volume form on** $\mathbb{R}^3$**.** Suppose x^1, x^2, x^3 are Cartesian coordinates on $\mathbb{R}^3$ and ϵ_{ijk} is the Levi-Civita symbol. (a) Verify that $\Omega = \frac{1}{3!}\epsilon_{ijk} dx^i \wedge dx^j \wedge dx^k = dx^1 \wedge dx^2 \wedge dx^3$. (b) Transform this 3-form to spherical polar coordinates $x^1 = r\sin\theta\cos\phi$, $x^2 = r\sin\theta\sin\phi$, $x^3 = r\cos\theta$ and find the components of Ω in the new variables. (c) For a suitable function f, express the volume form as $\Omega = f(r,\theta,\phi) dr \wedge d\theta \wedge d\phi$ and compare f with the determinant of the Jacobian.

2. $\langle \mathbf{2+4+1} \rangle$ **Pushforward and pullback.** Consider the smooth map $\phi : \mathbb{R} \to \mathbb{R}^2$ taking $x \mapsto (x, 0)$ in Cartesian coordinates. (a) Find the pushforward $\phi_* v$ of the vector field $v = x\partial_x$ on $\mathbb{R}$. (b) Find the pullbacks $\phi^*\alpha$ and $\phi^*\beta$ of the 1-forms $\alpha = dx$ and $\beta = dy$ on $\mathbb{R}^2$ as well as of the 2-form $\alpha \wedge \beta$. (c) What about the pullback of any 2-form on $\mathbb{R}^2$?