Mathematical Methods, Spring 2024 CMI

Assignment 5

Due by the beginning of the class (1030 am) on Tue, Feb 13, 2024

Differential forms

1. $\langle \mathbf{2+3+3} \rangle$ Consider the 1-form on $\mathbb{R}^2$, $\omega(x,y) = y\,dx - x\,dy$. Since $d(df) = d^2 f = 0$, an 'integrability' condition for ω to be exact ($\omega = d\sigma$ for some σ) is that it be closed ($d\omega = 0$). On $\mathbb{R}^2$, $d\omega = 0$ is also sufficient for ω to be exact. (a) Show that ω does not satisfy the integrability condition. (b) Look for an integrating denominator $\tau(x,y)$ and function $\sigma(x,y)$ such that $\omega/\tau = d\sigma$ is exact on some open subset of $\mathbb{R}^2$. (c) Analyze to what extent τ is unique. Remark: For a class of 1-forms ω associated to the infinitesimal heat added reversibly to a system, the 2^{nd} law of thermodynamics guarantees the existence of an integrating denominator τ, which is the absolute temperature. σ is the entropy function on the thermodynamic state space.

2. $\langle \mathbf{6} \rangle$ Suppose $A_\mu = (\phi, -\boldsymbol{A}) = (\phi, -A_x, -A_y, -A_z)$ are the Cartesian components of the gauge potential 1-form $A = A_\mu dx^\mu$ on 4d space-time with coordinates $x^\mu = (ct, x, y, z)$. Here ϕ and $\boldsymbol{A}$ are the scalar and vector potentials. Find the components of the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, arrange them in a 4×4 matrix and express them in terms of the components of the electric and magnetic fields $\boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{A} - \nabla\phi$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$.