Mathematical Methods, Spring 2024 CMI

Assignment 4

Due by the beginning of the class (1030 am) on Tue, Feb 6, 2024

1-forms and 2-forms

1. $\langle \mathbf{2}+\mathbf{2}+\mathbf{2} \rangle$ Suppose f and g are two smooth real-valued functions on 3d Euclidean space $\mathbb{R}^3$ with Cartesian coordinates x, y, z. (a) Find the components of $df \wedge dg$ in the coordinate basis for 2-forms. (b) Compare with the components of the cross product $\nabla f \times \nabla g$ familiar from vector calculus. (c) What is the geometric meaning of $\nabla f \times \nabla g$?

2. $\langle \mathbf{3}+\mathbf{3} \rangle$ Consider the 1-form $\alpha = (x dx + y dy)/r^2$ on the punctured x-y plane $\mathbb{R}^2 \setminus (0,0)$ where $r^2 = x^2 + y^2$. (a) Show that the exterior derivative $d\alpha = 0$. We say that α is a closed form 1-form. (b) Find a smooth function $f(x, y)$ on the punctured plane such that $\alpha = df$. If there is such an f, we say that α is an exact differential form.

3. $\langle \mathbf{2} \rangle$ Suppose $A = A_\mu dx^\mu$ is the gauge potential 1-form on 4d space-time with coordinates $x^\mu = (ct, x, y, z)$. Define the field strength 2-form $F = dA$. Find the components $F_{\mu\nu}$ of the field strength two form $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$.