Mathematical Methods, Spring 2024 CMI

Assignment 3

Due by the beginning of the class (1030 am) on Tue, Jan 30, 2024

Vector fields

1. ⟨**8**⟩ Suppose $u = u^i \partial_i$, $v = v^j \partial_j$ and $w = w^k \partial_k$ are three vector fields expressed in local coordinates $x^1, \cdots, x^n$ on an n-dimensional manifold. Use the formula for the commutator of vector fields to show that they satisfy the Jacobi identity $[u,[v,w]]+[w,[u,v]]+[v,[w,u]] = 0$.

2. ⟨**3 + 3**⟩ (a) Show that the Lie derivative $\mathcal{L}_u v = [u,v]$ satisfies the Leibniz rule: $\mathcal{L}_u(fv) = (\mathcal{L}_u f)v + f\mathcal{L}_u v$ for a smooth function f. (b) Is $[fu,v] = f[u,v]$? What can be said about $\mathcal{L}_{fu} v$? (c) Is the commutator linear over the space of functions? Why?

3. ⟨**4+4**⟩ Consider the vector field on the Euclidean plane given by $v = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$. (a) Sketch the vector field on the x-y plane by displaying arrows. Which way does it point at the origin? (b) Solve for the integral curve $(x(t), y(t))$ through the point $(x(0) = 1, y(0) = 0)$ and plot it. Compare with the previous sketch and comment on whether the arrows are tangent to the integral curve.