Mathematical Methods, Spring 2024 CMI

Assignment 1

Due by the beginning of the class on Jan 9, 2024

Charts and transition functions

1. $\langle \mathbf{3+3+3+3+3} \rangle$ Consider the unit sphere S^2 embedded in $\mathbb{R}^3$: $x^2+y^2+z^2=1$. Let $N=(0,0,1)$ be the North pole of the sphere and consider the equatorial plane E defined by $z=0$. Given any point $(x,y,z) \neq N$ on S^2, we define its **stereographic projection** to be the unique point $(X,Y) \in E$ through which the line joining N and P passes (see Fig. 1). (a) Describe how you might define the image of N under the stereographic projection using suitable limits. (b) Express the coordinates (X,Y) of the stereographic projection of P in terms of x,y,z. The stereographic projection from the North pole provides a coordinate chart on $S^2 \setminus N$ (sphere with N excluded). (c) Similarly, the stereographic projection from the South pole $S=(0,0,-1)$ to the equatorial plane defines a coordinate chart on $S^2 \setminus S$. Find the coordinates (X',Y') of the point $P=(x,y,z) \in S^2$ of the stereographic projection from S. (d) Find the transition function that expresses (X',Y') in terms of (X,Y) on the overlap $S^2 \setminus \{S,N\}$ between the two coordinate charts. (e) Is the transition function smooth? Why?

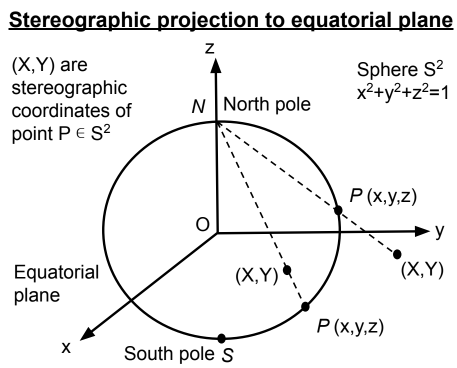

Figure 1: Stereographic coordinates (X,Y) of a point P on the sphere S^2 are given by the point of intersection with the equatorial plane of the line from the North pole through P.