Notes on Fluid Dynamics, CMI, Autumn 2024

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Course website http://www.cmi.ac.in/~govind/teaching/fluid-dyn-o24

Contents

1 Introduction to mechanics of deformable media

Continuum mechanics begins by dealing with the nonrelativistic classical dynamics of continuous deformable media. Examples are oscillations of stretched strings, heat conduction in rods, elastic motion of solids (rods/beams), motion of fluids[1] (air, water) and plasmas (ionized gases), in roughly increasing order of complexity. All of these systems involve a very large number of molecules (or degrees of freedom) and we will treat them as continuous mass/charge distributions with an infinite number of degrees of freedom. Thus, unlike particle mechanics, continuum mechanics

[1]Collectively, sand grains sometimes flow like a fluid, though individual grains display properties normally associated with a solid.

deals with fields. Examples of fields include the height of a stretched string, temperature, elastic displacement, mass density, velocity, pressure, internal energy, specific entropy, charge density, current density, electric and magnetic fields. While a classical point particle is *somewhere* at any given time, a classical field is *everywhere* at any given instant! Thus, continuum mechanics is a collection of (primarily nonrelativistic, classical) field theories. Electromagnetism and gravitation are other examples of field theories, though they often involve relativistic and/or quantum effects.

Due to the larger number of degrees of freedom, the dynamics of deformable bodies is generally more complicated than that of point particles or rigid bodies. In fact, we can imagine a rigid body becoming deformable by relaxing the constraints that fix the distances between its constituents.

There are two principal formalisms for treating mechanics of continuous media, the so-called Lagrangian and Eulerian descriptions. The former is closer to our treatment of systems of particles: we follow the motion of each molecule or fluid element (to be defined in Sect. 3) or bit of string. For example, if a fluid element occupied the location $\boldsymbol{a}$ at $t = 0$, then we seek the trajectory $\boldsymbol{r}(\boldsymbol{a}, t)$ of this fluid element, which should be determined by Lagrange's equations (ironically, this treatment was originally attempted by Euler). The Lagrangian description is particularly useful if we have some way of keeping track of which material element is where. This is usually not possible in a flowing liquid or gas, but is possible in a vibrating string since the bits of string are ordered and may be labeled by their location along the string or by their horizontal coordinate x for small vertical vibrations of a string that does not 'bend over'. For an elastic solid, the corresponding variable is the local displacement field $\boldsymbol{s}(\boldsymbol{r}, t)$ or $\boldsymbol{\xi}(\boldsymbol{r}, t)$ which represents the departure from the equilibrium location of the element that was originally at $\boldsymbol{r}$. In a fluid like air or water, it is difficult to follow the motion of individual fluid elements due to the tendency to mix.

So Euler developed the so-called Eulerian description, which attempts to understand the dynamics of quantities (Eulerian variables) such as density $\rho(\boldsymbol{r}, t)$, pressure $p(\boldsymbol{r}, t)$, velocity $\boldsymbol{v}(\boldsymbol{r}, t)$ and temperature $T(\boldsymbol{r}, t)$ in a fluid at a specified *observation point* $\boldsymbol{r}$ at time t. However, it must be emphasized that the laws of mechanics (Newton's laws) apply to material particles or fluid elements, not to points of observation, so one must reformulate the equations of motion so that they apply to the Eulerian variables. The equations of motion in continuum mechanics are invariably expressed as partial differential equations for fields (such as the density of a fluid or height of a string at a given location and time). Thus, we are dealing with the classical dynamics of fields. We will now discuss the flow of fluids, primarily from an Eulerian perspective.

2 Introduction to fluid mechanics

Fluid flows are all around us: the air through our nostrils, tea stirred in a cup, water down a river and charged particles in the ionosphere. The flow of fluids can be fascinating to watch. It is also an interesting branch of physics to which many of the best scientists from the early days of Leonardo da Vinci, Isaac Newton, Daniel Bernoulli and Leonhard Euler have contributed. Fluid dynamics finds application in numerous areas: flight of airplanes and birds, weather prediction, blood flow in the heart and blood vessels, waves on the beach, ocean currents and tsunamis, flows in the molten metallic core of the Earth, controlled nuclear fusion in a tokamak, jet engines in rockets, motion of charged particles in the solar corona and astrophysical jets, accretion disks around active galactic nuclei, formation of clouds, melting of glaciers, climate change, sea level rise, traffic flow, building pumps and dams, etc. Fluid flows can range from regular and predictable (laminar) to seemingly disorganized and unpredictable (turbulent) while displaying remarkable patterns.

We believe[2] we know the macroscopic physical laws governing fluid motion. In the absence of dissipation, they are the local conservation laws of mass, momentum and energy along with a thermodynamic equation of state. The resulting equation for the flow velocity in 'ideal' (dissipationless) flow goes back to the work of Euler (1757). In the presence of dissipation (viscosity, thermal conductivity, etc.), local conservation of mass continues to hold although the ideal momentum and energy equations are modified (in the simplest possible way) using empirical macroscopic laws of Newton and Fourier governing diffusion of momentum and heat to arrive at the equations for viscous flow. The corresponding equation for the flow velocity was introduced by Claude-Louis Navier (1822) and George Gabriel Stokes (1845). It is important to bear in mind that these equations of macroscopic fluid mechanics were postulated based on empirical observations, macroscopic conservation laws and the principles of minimality and simplicity rather than by a direct application of Newton's second law to individual molecules. In fact, these equations were proposed well before the molecular structure of matter was established. What is more, although we now know the laws of molecular dynamics accurately, it has not been possible to rigorously deduce the equations of fluid mechanics from them[3]. In this framework, the equations of fluid mechanics have to be validated by comparing their predictions[4] with macroscopic experimental measurements and observations. Fortunately, in many cases where such comparisons have been possible, there is evidence in favor of the fluid equations. However, there are situations where one needs to modify them (e.g., to account for a nonlinear stress-strain relation or the polymeric structure of constituent molecules) or abandon them (e.g., when one is interested in phenomena on molecular length scales).

[2] Unlike in the application of Newton's laws to a pair of point particles or a rigid body, there are significant approximations, imprecise notions of averaging and plausible assumptions involved in arriving at the equations governing macroscopic fluid motion.

[3] Well after their formulation, some of these macroscopic fluid equations (especially for dilute gases, but not for liquids) have been shown (by L Boltzmann, S Chapman, D Enskog and others) to follow from the molecular kinetic theory of gases through a coarse-graining procedure based on some plausible assumptions and approximations. In this chapter, we will introduce the equations of fluid mechanics from a macroscopic viewpoint and make no attempt to derive them from kinetic theory.

[4] As in the rest of continuum mechanics, the evolution equations of fluid dynamics are partial differential equations. However, these equations are nonlinear and despite much progress since the time of Euler, Navier and Stokes, it is still a challenge to calculate (even with the best of computers) many features of commonly occurring flows.

3 Fluid element, local thermal equilibrium and dynamical fields

In a fluid description, we do not follow the microscopic positions and velocities of individual molecules. We focus instead on macroscopic fluid variables such as velocity, pressure, density, energy and temperature that we assign to a *fluid element* by averaging over it. By a fluid element (sometimes called a material element), we mean a sufficiently large collection of molecules so that concepts such as 'volume occupied' make sense and yet small in extent compared to the macroscopic length scales of phenomena we wish to describe. Thus, quantities such as the density and velocity will be assumed not to vary appreciably over a fluid element. For example, we could divide a bucket with about 10^{23} molecules into 10^3 fluid elements, each containing 10^{20} molecules. Thus, we model a fluid as a continuum system with an infinite number of degrees of freedom[5]. The fluid description applies to phenomena on length scales large compared to the typical mean free path between collisions of molecules. On shorter length scales, the fluid description breaks down[6], though Boltzmann's kinetic theory of molecules applies.

A flowing fluid is generally *not* in global thermal equilibrium. What this means is that it may not be possible to assign a common temperature to all parts of a fluid, and heat could be transported between parts of a fluid. Nevertheless, collisions between molecules typically establish local thermodynamic equilibrium so that we may assign a local absolute temperature T, pressure p and density ρ to fluid elements, satisfying an equation of state (such as that of an ideal gas[7] $p = \rho RT/\mu$). Sometimes, it is convenient to replace some of these thermodynamic state variables with specific entropy s (entropy S per unit mass) or specific internal energy ϵ (energy per unit mass) or specific volume $v = 1/\rho$. Each of these quantities could vary from one fluid element to another and also with time. From an Eulerian standpoint, at each location $\boldsymbol{r}$ in a fluid at time t, we have the dynamical fields of density $\rho(\boldsymbol{r}, t)$, pressure $p(\boldsymbol{r}, t)$, specific entropy $s(\boldsymbol{r}, t)$, temperature $T(\boldsymbol{r}, t)$, etc. In addition to these scalar fields, we have the velocity vector field $\boldsymbol{v}(\boldsymbol{r}, t)$ that is instantaneously tangent to the flow at each point $\boldsymbol{r}$.

4 Fluid statics: aero- or hydrostatics

Before considering fluid flows in more detail, we briefly remark upon the special situation that prevails when the fluid is not in motion in the frame considered. This is usually called hydrostatics or sometimes aerostatics (if one wishes to emphasize that the density is inhomogeneous). In fluid static equilibrium, each fluid element is at rest due to a balance between surface and body forces. Surface forces are those that act on the element across its boundary due to material just outside the surface. The most

[5]To specify the pattern of a flow we must, among other things, specify the fluid velocity at each of the infinitely many points in the container.

[6]In going from a molecular description to a fluid description, we replace sums over individual molecules by integrals over the region occupied by the fluid, with fluid elements roughly playing the role of infinitesimal integration elements. A system with a very large but finite number of molecular degrees of freedom is approximated by a continuum system with infinitely many degrees of freedom.

[7]Here $R = 8.314$ Joules per Kelvin per mole is the universal gas constant and μ the molar mass, 12 grams per mole for Carbon-12

common body force is gravity, which acts over the whole volume of the fluid element. To obtain the equations of hydrostatic equilibrium, we consider a small fluid element of mass $\delta m = \rho\, \delta V$ occupying a volume δV. The external body force such as gravity acting on the fluid element is $\boldsymbol{f} \delta V$ where $\boldsymbol{f}$ is the body force per unit volume ($\boldsymbol{f} = \rho \boldsymbol{g}$ for gravity, where $\boldsymbol{g}$ is the acceleration vector due to gravity). In addition, we have the surface force due to the pressure exerted on the fluid element by the fluid surrounding the element. To calculate this, assume the fluid element is an infinitesimal cuboid with sides of length dx, dy and dz.

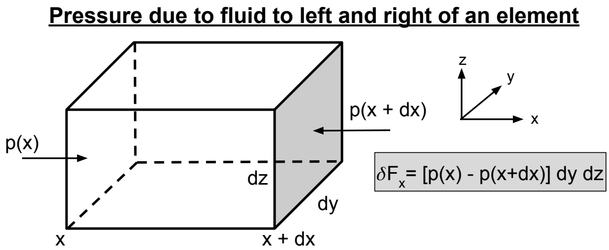


Figure 1: Horizontal force on a fluid element due to material to the left and right.

As shown in Fig. 1, the net pressure force in the $\hat{x}$ direction is the product of the area $dydz$ and pressure difference between the left and right faces: $\delta F_x \approx -\frac{\partial p}{\partial x} dx \times dydz$. The negative sign is because pressure tends to compress the element and the net force is leftward if p on the right face is larger than on the left face. Thus, the total pressure force on the fluid element is

$$\delta \boldsymbol{F}_{\text{pressure}} = \delta F_x \hat{x} + \delta F_y \hat{y} + \delta F_z \hat{z} = -(\boldsymbol{\nabla} p)\, \delta V. \tag{1}$$

For the element to be in static equilibrium, we must have

$$\boldsymbol{f} - \boldsymbol{\nabla} p = 0 \quad \text{or} \quad \boldsymbol{\nabla} p = \rho \boldsymbol{g}. \tag{2}$$

where $\boldsymbol{f}$ and $\boldsymbol{g}$ are the body forces per unit volume and mass respectively. This is one equation for two unknown functions, the pressure and density. It is usually supplemented by an ‘equation of state’ relating pressure to density. For an incompressible liquid, ρ can often be assumed to be a constant. For an ideal gas at a fixed temperature T, the equation of state is $p = \rho RT/\mu$ where μ is the molar mass and R the universal gas constant. This is usually written as Boyle’s law $(p/p_o) = (\rho/\rho_o)$ where p_o is the pressure at a reference density ρ_o. If the pressure and density variations are at constant entropy (reversible adiabatic process) rather than constant temperature, the corresponding formula is $(p/p_o) = (\rho/\rho_o)^\gamma$ where the adiabatic index $\gamma = C_p/C_v$ is the ratio of heat capacities at constant pressure and volume. □

Example: Atmospheric pressure. For example, let us find the density and pressure as a function of height z in the atmosphere, assuming it is in aerostatic equilibrium and treating the temperature and acceleration due to gravity as independent of height. The force balance equation reduces to

$$\frac{\partial p}{\partial z} = -g\rho(z) \quad \text{or} \quad \frac{dp}{p} = -\frac{g\rho_o}{p_o} \quad \Rightarrow p(z) = p(0) e^{-\rho_o g z/p_o}. \tag{3}$$

Thus, the pressure and density decrease exponentially with height if we ignore the temperature and gravity variations. Prob. **??** treats this aerostatic situation with the isentropic equation of state $p \propto \rho^\gamma$, which is more realistic. □

A frequently encountered circumstance is one where the body force field per unit mass is the (negative) gradient of a potential $\boldsymbol{g} = -\boldsymbol{\nabla}\varphi$. Such a force is called *conservative*. Then $\boldsymbol{\nabla} p = -\rho \boldsymbol{\nabla}\varphi$. If, moreover, the density is a constant, we have $\boldsymbol{\nabla}(\frac{p}{\rho} + \varphi) = 0$. So $p/\rho + \varphi$ must be a constant. In particular, an equipotential surface must also be a surface of constant pressure (an isobar). For example, the free surface of a liquid is an isobar (pressure equal to atmospheric pressure), and hence must also be an equipotential surface within these approximations.

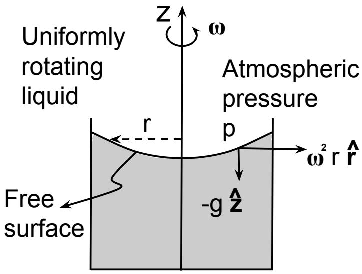


Figure 2: Parabolic free surface of a uniformly rotating liquid.

Example: Free surface of rotating liquid. Let us apply (2) to determine the shape of the free surface of a liquid that is rotated at a constant angular velocity $\omega\hat{z}$ in a bucket (cf. Fig. 2). After some time, the surface of the liquid is found to reach an equilibrium shape. In a corotating frame, the body forces per unit mass are gravity $-g\hat{z}$ and the centrifugal force $r\omega^2\hat{r}$ where we use cylindrical coordinates r, θ, z. Thus, the body force per unit mass is the negative gradient of the effective potential $\varphi = gz - \frac{1}{2}r^2\omega^2$. Once the liquid settles into equilibrium, $p/\rho + gz - \frac{1}{2}\omega^2 r^2$ is a constant. On the free surface, the pressure is constant, equal to atmospheric pressure. So the equation for the free surface $gz - \frac{1}{2}\omega^2 r^2 =$ constant, describes a paraboloid obtained by rotating the parabola $gz - \frac{1}{2}\omega^2 x^2 =$ constant, about the z axis. □

5 Flow visualization: streamlines, pathlines and streaklines

In fluid mechanics, when we speak of the velocity of a flow, we are referring not to the random thermal motions of individual molecules, but to the velocity of the overall flow. The latter is smoother since an average over molecules in each fluid element has been performed to arrive at the flow velocity field.

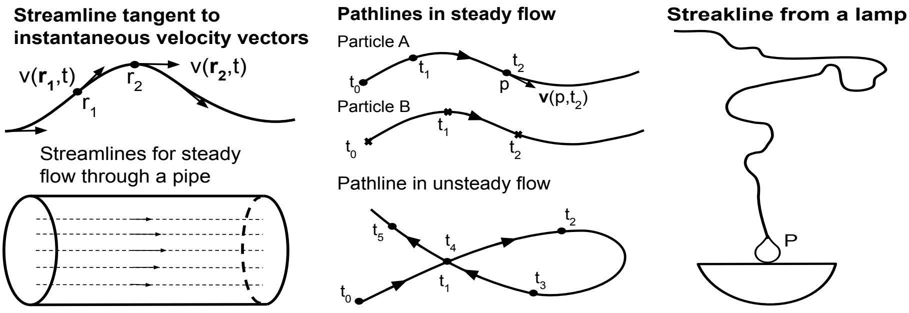
If the velocity vector field at every point of observation is independent of time, we say the velocity field is steady, $\boldsymbol{v}(\boldsymbol{r}, t) = \boldsymbol{v}(\boldsymbol{r})$. More generally, we will say that a fluid flow is steady if the velocity, density, pressure, temperature, specific entropy, etc., are independent of time at every point of observation. To aid in the visualization of a flow we define the concepts of streamlines, streaklines and pathlines. All three coincide for a steady flow, though not in general. For steady flow, they are the 'field lines' or integral curves of the velocity vector field, i.e., curves that are everywhere tangent to $\boldsymbol{v}(\boldsymbol{r})$ (see Fig. 3a). They are the trajectories of test particles moving in the steady flow, i.e., solutions of the ODEs and initial conditions

$$\frac{d\boldsymbol{r}}{ds} = \boldsymbol{v}(\boldsymbol{r}(s)) \quad \text{and} \quad \boldsymbol{r}(s_o) = \boldsymbol{r}_o. \tag{4}$$

Here, s is the parameter along the integral curve, it is the time that parametrizes the trajectory of the test particle moving in the steady flow. If we write these in Cartesian components $\boldsymbol{r}(s) = (x(s), y(s), z(s))$ and $\boldsymbol{v}(\boldsymbol{r}) = (v_x(\boldsymbol{r}), v_y(\boldsymbol{r}), v_z(\boldsymbol{r}))$, then the ODEs for field lines become

$$\frac{dx}{ds} = v_x, \quad \frac{dy}{ds} = v_y \quad \text{and} \quad \frac{dz}{ds} = v_z \quad \text{or} \quad \frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z} = ds. \tag{5}$$

Streamlines. More generally, consider a possibly nonsteady flow. Streamlines at the observation time t_o are defined as the integral curves of the velocity field $\boldsymbol{v}(\boldsymbol{r}, t_o)$. The streamline through any point of observation P with position vector $\boldsymbol{r}(P)$ at a given time t_o is tangent to the velocity vector $\boldsymbol{v}(\boldsymbol{r}(P), t_o)$. At a given instant of time, streamlines cannot intersect. Since the flow may not be steady, the streamlines will in general change with time. Streamlines of the velocity field are analogous to the field lines of a (generally time-dependent) electric or magnetic field. In particular, for a divergence-free ($\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$) flow, streamlines cannot emerge or spread out from a point or region, just as magnetic field lines cannot. A flow that is spatio-temporally regular is called laminar. An example is the slow, steady flow of water through a pipe, where streamlines are parallel as in Fig. 3a.



(a) (b) (c)

Figure 3: (a) Streamlines encode the instantaneous flow pattern. (b) Pathline of a speck of sawdust as it is carried by a flow. (c) Caricature of a streakline in the air above a lamp's burning wick at the point P. The burning wick introduces particles of soot into the air, which are carried by the air flow. The curve along which the soot lies at a given time is the instantaneous streakline. A burning incense stick also produces a streakline if we ignore the slow movement of the point of injection (reduction in length of the stick as it burns).

Streamlines have information on the current flow. For example, we could draw the streamlines of the monsoon winds over the Indian peninsula at the onset of the South-

West monsoon on June 5, 2012. These streamlines changed with time and partly reversed direction during the ‘receding’ North-East monsoon in November 2012.

Pathlines are the trajectories of individual fluid particles. For example, if we introduced a small speck of saw dust[8] (which reflects light) into the fluid and took a movie of its trajectory, we would get its pathline (see Fig. 3b). At any point P along a pathline, it is tangent to the velocity vector at P at the time the particle passed through P. Pathlines can intersect themselves or even retrace themselves, for instance if a fluid particle goes round and round in a container. Two pathlines can intersect if the point of intersection corresponds to a different time on each of the two trajectories. For example, two different dust particles may pass through the same point in a room on two different days.

Streaklines. Suppose a small quantity of dye is continuously injected into a fluid flow at a fixed point of injection P. The dye is so chosen that the dye particles do not diffuse in the fluid. Rather, a dye particle tends to stick to the first fluid particle it encounters and flows along with it. So the dye released at time t sticks to the fluid particle that passes through P at time t and is then carried by that particle. The resulting highlighted curve is the *streakline* through P as illustrated in Fig. 3c. So at a given time of observation $t_{\rm obs}$, a streakline is the locus of all current locations of particles that passed through P at some time $t \le t_{\rm obs}$ in the past. Unlike streamlines, streaklines provide information on the history of the flow. Streaklines for a given flow are governed by three quantities: the point of injection P, the time of observation $t_{\rm obs}$ and the time when the injection of dye began t_i. Such a streakline always begins at P and extends to a point determined by t_i when injection began. In practice, streaklines get blurred by diffusion of the dye in the fluid, however they are reasonably sharp for a time short compared to the diffusion time scale. A streakline cannot self-intersect.

6 Material derivative

In the Eulerian description of fluid motion, we are interested in the time development of various fluid dynamical variables such as velocity, pressure, density and temperature at a given point of observation $\boldsymbol{r} = (x, y, z)$ in the container. This is reasonable if we are interested in predicting the weather changes at the point of observation over the course of time. For instance, the change in density at a fixed location is $\frac{\partial \rho(\boldsymbol{r})}{\partial t}$. However, different fluid particles will arrive at the point $\boldsymbol{r}$ as time passes. It is also of interest to know how the corresponding dynamical variables evolve, not at a fixed location but for a fixed small fluid element, as in a Lagrangian description. This is especially important since the dynamical laws of mechanics apply directly to the fluid particles, not to the point of observation. So, we may ask how a variable changes along the flow, so that the observer is always attached to a fixed fluid element (or ‘material element’) and travels along its pathline. For instance, the change in density of a

[8] Leonardo da Vinci (1452-1519) suspended fine sawdust in water and observed the motion of the saw dust as it was carried by the flow. By contrast, pollen grains were used by Robert Brown (1827) to indirectly reveal the random thermal motion of molecules under a microscope.

fluid element in a small time dt as it moves from location $\boldsymbol{r}$ to $\boldsymbol{r} + d\boldsymbol{r}$ is

$$d\rho = \rho(\boldsymbol{r} + d\boldsymbol{r}, t + dt) - \rho(\boldsymbol{r}, t) \approx d\boldsymbol{r} \cdot \boldsymbol{\nabla}\rho + \frac{\partial \rho}{\partial t}\, dt. \tag{6}$$

We divide by dt, take the limit $dt \to 0$ and observe that $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$ is the velocity of the fluid at the point $\boldsymbol{r}$ at time t. Thus, the instantaneous rate of change of density of a fluid element that is located at $\boldsymbol{r}$ at time t is

$$\frac{D\rho}{Dt} \equiv \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\rho = (\partial_t + v_x\partial_x + v_y\partial_y + v_z\partial_z)\,\rho. \tag{7}$$

$\frac{D}{Dt} = \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}$ is called the material[9] (also total, substantial, convective) derivative. It can be used to express the rate of change of a physical quantity (velocity, pressure, temperature, etc.) associated to a fixed fluid element, i.e., along the flow specified by the velocity field $\boldsymbol{v}$. This formula for the material derivative bears a resemblance to the rigid body formula relating the time derivatives of a vector relative to the lab and corotating frames: $\left(\frac{d\boldsymbol{A}}{dt}\right)_{\text{lab}} = \left(\frac{d\boldsymbol{A}}{dt}\right)_{\text{rot}} + \boldsymbol{\Omega} \times \boldsymbol{A}$. A quantity f (could be a scalar or a vector) is said to be conserved along the flow or dragged by the flow if its material derivative vanishes $\frac{Df}{Dt} = 0$.

Since $\frac{D}{Dt}$ is a first order partial differential operator, Leibniz's product rule of differentiation holds for scalar functions f, g: $\frac{D(fg)}{Dt} = f\frac{Dg}{Dt} + \frac{Df}{Dt}g$. Similarly, for a scalar f and vector field $\boldsymbol{w}$, we check that the Leibniz rule holds

$$\frac{D(f\boldsymbol{w})}{Dt} = \frac{Df}{Dt}\boldsymbol{w} + f\frac{D\boldsymbol{w}}{Dt}. \tag{8}$$

7 Compressibility, incompressibility and divergence of velocity field

We define a flow to be incompressible if the volume occupied by any fixed fluid element[10] (not necessarily small) remains constant in time although its shape may change. This is approximately true for water flowing in a hose pipe. Generally, liquids tend to be incompressible, they offer a large opposing force to attempts to change volume. Gases are more compressible, and high speed flows in gases tend to be compressible. However, the same material (like air) under different conditions may behave differently, depending on the speed of the flow in comparison to the speed of sound, as we will explain later in this section.

To clarify the idea of incompressibility, we ask how the volume V of a region Ω occupied by a fluid changes with time[11], i.e., we seek an expression for $\frac{dV}{dt}$. Suppose Ω is bounded by a surface $S = \partial\Omega$ with outward area element $d\boldsymbol{S}$ and outward unit

[9] The adjectives *material* or *substantial* are meant to convey that D/Dt is a rate of change computed while moving with the material or substance.

[10] By a fixed fluid element we mean a fixed collection of molecules. One can think of them as being surrounded by an imaginary impermeable membrane that instantaneously assumes the shape of the region they occupy.

[11] Here, $\frac{dV}{dt}$ is not the material derivative in the strict sense of Sect. 6, since V is not a local field. However, it is similar in spirit as it is the rate of change of volume following the flow.

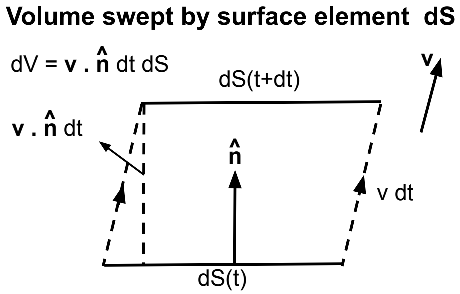


Figure 4: Surface element $d\boldsymbol{S} = \hat{\boldsymbol{n}}\, dS$ is carried by a flow $\boldsymbol{v}$ over a time dt sweeping out a volume $dV = \boldsymbol{v} \cdot \hat{\boldsymbol{n}}\, dt\, dS$. The figure shows a side view of the volume.

normal $\hat{\boldsymbol{n}}$ such that $d\boldsymbol{S} = \hat{\boldsymbol{n}}\ dS$. In a small time dt, the region Ω changes by a movement of its bounding surface[12] in the direction of $\boldsymbol{v}$. At a point $\boldsymbol{r}$ on $\partial\Omega$, the surface element dS moves out a perpendicular distance $\boldsymbol{v} \cdot \hat{\boldsymbol{n}}\ dt$ where $\boldsymbol{v}$ is the fluid velocity at the point $\boldsymbol{r}$ (see Fig. 4). Thus, the change in volume $dV(dS)$ due to the area element dS moving out a bit is $\boldsymbol{v} \cdot \hat{\boldsymbol{n}}\ dt\ dS$. To include the contributions of all area elements, we integrate over the entire bounding surface to arrive at

$$\frac{dV}{dt} = \int_S \boldsymbol{v} \cdot \hat{\boldsymbol{n}}\, dS = \int_\Omega \boldsymbol{\nabla} \cdot \boldsymbol{v}\ d\boldsymbol{r}. \tag{9}$$

The last equality uses Gauss' divergence theorem to transform the surface integral into a volume integral. Since this is true for a fluid parcel of any volume (above molecular sizes), let us specialize to a small fluid element Ω (so that $\boldsymbol{\nabla} \cdot \boldsymbol{v}$ is roughly constant over its extent) at location $\boldsymbol{r}$ having volume δV. Then,

$$\frac{d\delta V}{dt} = \frac{D\,\delta V}{Dt} \approx (\boldsymbol{\nabla} \cdot \boldsymbol{v})\,(\delta V) \quad \text{or} \quad \boldsymbol{\nabla} \cdot \boldsymbol{v} = \lim_{V \to 0} \frac{1}{V}\frac{dV}{dt} = \lim_{V \to 0} \frac{d \log V}{dt}. \tag{10}$$

So the divergence of the velocity field is the fractional rate of change of volume of a small fluid element.

A flow is incompressible if each fluid element maintains its volume during the flow, i.e., $\frac{dV}{dt} = 0$ for all V (above molecular scales). It follows that a flow is incompressible iff the velocity field is divergence-free: $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$.

Examples. A simple example of an incompressible flow is one where the density of the fluid is the same everywhere and at all times. In fact, if the density ρ is a constant, then the volume of an element is a fixed multiple $(1/\rho)$ of its mass. However, the mass of a *material* element is conserved, so its volume must remain constant. A more general example of an incompressible flow is one where the density of a given fluid element is constant in time, though different fluid elements may have different densities. This happens for horizontal flows in the atmosphere, where the density is

[12] We neglect the infinitesimal change in the area dS of the surface element due to the flow. The change in volume due to such a change is of second order in infinitesimals. The surface area of a material element can change even in incompressible flow.

stratified by height though the flow is horizontal. Note that the same fluid (e.g., air) under different conditions may exhibit incompressible and compressible flows. The study of compressible flows is usually termed gas dynamics or aerodynamics, while the study of incompressible flows is often termed hydrodynamics.

Compressibility and bulk modulus. Incompressibility means the volume of a fluid element does not change irrespective of the pressure applied across its surface. A measure of the compressibility[13] of a flow is the *compressibility* $\kappa = -\frac{1}{V}\frac{\partial V}{\partial p}$. The negative sign ensures that $\kappa \geq 0$, since pressure tends to decrease volume in most materials. Thus, $\kappa \to 0$ in an incompressible flow.

The reciprocal of compressibility is called the bulk modulus K. Since the mass of a fluid element is conserved, incompressibility may be taken to mean that the density does not change with applied pressure. Indeed, since $\rho \propto 1/V$, we may write the compressibility[14] also as $\kappa = \frac{1}{\rho}\frac{\partial \rho}{\partial p}$. □

Relation to speed of sound. Intuitively, a sound wave is a wave of compression and expansion. As we will learn in Sect. 12, a sound wave propagates changes in density and travels at the speed c_s where $c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$ (for flow with constant specific entropy s). Evidently, c_s grows as the compressibility κ decreases. Solids tend to be less compressible than gases. As a consequence, sound propagates faster in steel than in air and we can hear an approaching train on a railway track earlier than it is heard through the air. If the flow velocity $|\boldsymbol{v}|$ is small compared to the speed of sound c_s, then the flow can usually be approximated as incompressible[15]. In fact, we may regard a strictly incompressible flow as one where the speed of sound is infinite. Crudely, any attempt by the flow to alter the density of a fluid element is immediately wiped out since sound travels much faster than the flow and irons out the change. □

Incompressibility in 2d: stream function. The condition for a vector field on the x-y plane to be incompressible can be solved in terms of a scalar *stream function* $\psi(x, y)$. Indeed, suppose $\boldsymbol{v} = (u(x, y), v(x, y), 0)$, then $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ becomes the condition $u_x + v_y = 0$, where subscripts denote partial derivatives. Now, if

$$u = \psi_y \quad \text{and} \quad v = -\psi_x, \tag{11}$$

then the incompressibility condition is identically satisfied. In 3d vector notation, we can regard $\psi(x, y)\hat{z}$ as a vector potential for the incompressible velocity field: $\boldsymbol{v} = \boldsymbol{\nabla} \times (\psi \hat{z})$, which is then automatically divergence-free. This is similar to how the solenoidal magnetic field is expressed in terms of a vector potential $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ in electrodynamics. □

[13] Intuitively, compressibility measures how much the volume of a fluid element decreases in response to a unit increase in applied pressure. To obtain a nontrivial limit as $V \to 0$, we divide by the volume V of the fluid element to arrive at the local (intensive) variable κ.

[14] In evaluating this partial derivative using the thermodynamic equation of state (see Sect. 10), a third variable such as temperature or entropy is held fixed. So one has slightly different notions of compressibility depending on what is held fixed.

[15] The Mach number $M = |\boldsymbol{v}|/c_s$ (which could depend on location and time) is a way of quantifying this. The Mach number is zero in incompressible flow. Flow in regions where $M < 1$ is called subsonic while it is supersonic where $M > 1$.

To sum up, we introduced the idea of incompressibility via the divergence of $\boldsymbol{v}$ and then discussed the physical meaning of compressibility in terms of the density ρ. Pleasantly, the divergence-free condition $\boldsymbol{\nabla}\cdot\boldsymbol{v}=0$ may be expressed in terms of the material derivative of ρ via the continuity equation, as we will see in Sect. 8.

8 Local conservation of mass: continuity equation

The total mass of fluid in a given fluid element remains constant in time, since material does not enter or leave the element. Consider a small fluid element of volume δV in the vicinity of the point $\boldsymbol{r}$ where the fluid density is $\rho(\boldsymbol{r})$ at time t. Then the mass of the fluid element is $\delta m=\rho\,\delta V$. The material derivative of δm must vanish. Using the Leibniz rule (8) and (10) we get for any small δV,

$$0=\frac{D\,\delta m}{Dt}=\frac{D(\rho\,\delta V)}{Dt}=\frac{D\rho}{Dt}\,\delta V+\rho\,\frac{D\,\delta V}{Dt}=\left(\frac{D\rho}{Dt}+\rho\boldsymbol{\nabla}\cdot\boldsymbol{v}\right)\delta V. \qquad (12)$$

Thus, we arrive at the *continuity equation* expressing conservation of mass

$$\frac{D\rho}{Dt}+\rho\,\boldsymbol{\nabla}\cdot\boldsymbol{v}=0. \qquad (13)$$

We immediately see that if the density is constant along the flow ($\frac{D\rho}{Dt}=0$), then the flow is divergence-free ($\boldsymbol{\nabla}\cdot\boldsymbol{v}=0$) and incompressible. Expanding the material derivative, we get

$$\frac{\partial\rho}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\rho+\rho\,\boldsymbol{\nabla}\cdot\boldsymbol{v}=0. \qquad (14)$$

In particular, if $\rho=\rho_0$ is constant in both time and space, then the flow must be incompressible. On the other hand, if the flow is incompressible, i.e., $\boldsymbol{\nabla}\cdot\boldsymbol{v}=0$, then the density must be constant along the flow, $\frac{D\rho}{Dt}=\frac{\partial\rho}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\rho=0$. We say the density is advected or transported by an incompressible flow.

Combining the last two terms in (14), the continuity equation can be written in local conservation form:

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})=0. \qquad (15)$$

We say that ρ is the locally conserved mass density and $\rho\boldsymbol{v}$ is the corresponding mass current density. The continuity equation says that the rate of change of density at a point is balanced by the divergence of the mass current density. We may also write (15) in integral form, by integrating over a region Ω that is *fixed* in space (does not move with the flow) and applying Gauss' divergence theorem:

$$\int_\Omega\frac{\partial\rho}{\partial t}\,d\boldsymbol{r}+\int_\Omega\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})\,d\boldsymbol{r}=0\quad\text{or}\quad\frac{d}{dt}\int_\Omega\rho\,d\boldsymbol{r}+\int_{S=\partial\Omega}\rho\boldsymbol{v}\cdot d\boldsymbol{S}=0. \qquad (16)$$

The 1^{st} term is the rate of increase of mass inside a fixed volume Ω. The 2^{nd} gives the outward flux of mass across the boundary S. So mass is neither created nor destroyed: it can only move around *continuously*, hence the name 'continuity' equation. If Ω is the entire flow domain, then the first term is the rate of increase of mass of the fluid as a whole, which must vanish provided the mass flux across the boundary is zero.

9 Euler equation for inviscid flow

An inviscid (sometimes called ideal) fluid flow is one where no resistance is offered to changes in shape that are not accompanied by a change in volume. We will elaborate on this shortly. In particular, ideal fluids assume the shape of the container; they lack a rigidity of form. This means that in an ideal flow, the force acting on a material element (anywhere in the fluid) across its surface, due to the material outside, is everywhere normal to the surface. Tangential surface forces tend to *shear* the element and change its shape without affecting its volume. On the other hand, normal surface forces tend to compress or expand[16] the element and thereby change its volume. The inward directed normal surface force per unit area is called pressure p. So in an inviscid flow, tangential or shearing stresses vanish irrespective of the location and orientation of the surface. In viscous flows, tangential forces typically arise between layers of fluid in relative motion. Thus, tangential forces are absent in hydrostatics.

Stress tensor[17]. In general, forces need not be either normal or tangential to surfaces[18] in the fluid, and they could vary in magnitude and direction with location. The stress tensor is a quantity that encodes the force per unit area acting across an element of surface. Let $\hat{\boldsymbol{n}}\,\delta S$ be a small surface element of area δS, with unit normal $\hat{\boldsymbol{n}}$, centered at $\boldsymbol{r}$. Let $\boldsymbol{F}(\hat{\boldsymbol{n}}\,\delta S, \boldsymbol{r})$ be the force that acts across the surface, its magnitude must be proportional to the area δS. Precisely, it is the force on the material on the side to which $\hat{\boldsymbol{n}}$ points, due to the material on the other side, as shown in Fig. 5. In general, $\boldsymbol{F}$ and $\hat{\boldsymbol{n}}$ point in different directions and are related by a linear transformation, the transformation of stress. If we choose to write all vectors in some basis, e.g., resolve them according to Cartesian components, then this linear relation may be written as

$$\boldsymbol{F}_i(\hat{\boldsymbol{n}}\,\delta S, \boldsymbol{r}) = \sum_j T_{ij}(\boldsymbol{r}) n_j\,\delta S. \tag{17}$$

The 3×3 matrix $T_{ij}(\boldsymbol{r})$ is called the stress tensor field. It depends only on the location $\boldsymbol{r}$ and not on the surface or $\hat{\boldsymbol{n}}$. By choosing a surface whose normal $\hat{\boldsymbol{n}}$ points in the j^{th} direction, we see that T_{ij} is then the i^{th} component of the force acting on the material towards the j^{th} direction of a surface of unit area whose normal points in the j^{th} direction. Alternatively, suppose δS is a small surface with normal $\hat{\boldsymbol{n}}$, then $\sum_j T_{ij} n_j (\delta S)$ is the i^{th} component of the force acting on the material on the side to which the normal $\hat{\boldsymbol{n}}$ points.

Example: stress tensor in hydrostatics and inviscid flow. By definition, hydrostatic pressure acts normal to any surface. So consider a small cuboid with axes along Cartesian axes. It follows that $T_{ij} = 0$ for $i \neq j$, as there are no tangential stresses. Moreover, $T_{33} = p$ since the force across the top surface (whose normal points along $\hat{z}$) due to the fluid below, is $p\hat{z}$. We get the same answer by considering the bottom surface. Proceeding in this way, $T_{ij} = p\delta_{ij}$. This formula for the stress tensor due to

[16]Normal surface forces that tend to expand an element are called tensile stresses, as is the case in an elastic rod that is being stretched. Tensile stresses correspond to a *negative* pressure.

[17]Here, we introduce the stress tensor in general, not necessarily for inviscid flow.

[18]These may be external or, more frequently, hypothetical internal surfaces in the fluid.

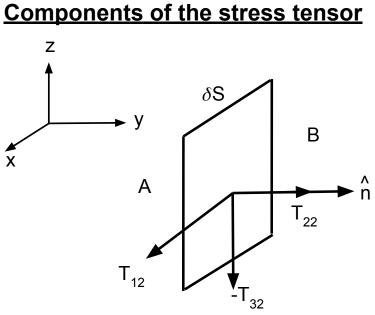


Figure 5: Components of the force due to fluid A on fluid B across a small surface with unit normal $\hat{\boldsymbol{n}}$ which here points along $\hat{y}$. T_{32} is the third component of the force on the material located on the second direction of the surface.

hydrostatic pressure is independent of basis: multiples of the identity matrix have the same components in any basis.

More generally, the absence of tangential stresses in an inviscid flow irrespective of orientation of surfaces implies that the stress tensor is diagonal in every basis, and must therefore be proportional to the identity: $T_{ij} = p\delta_{ij}$. □

Euler equation. To derive the equation of motion for an inviscid flow, consider a small fluid element of mass $\delta m = \rho\, \delta V$ occupying a volume δV and having instantaneous velocity $\boldsymbol{v}$. Let us write Newton's 2^{nd} law for this fluid element. The change in its velocity in a time dt as it is displaced from $\boldsymbol{r}$ to $\boldsymbol{r} + d\boldsymbol{r}$ is

$$d\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r} + d\boldsymbol{r}, t + dt) - \boldsymbol{v}(\boldsymbol{r}, t) \approx \frac{\partial \boldsymbol{v}}{\partial t} dt + (d\boldsymbol{r} \cdot \boldsymbol{\nabla})\boldsymbol{v}. \tag{18}$$

Dividing by dt, letting $dt \to 0$ and noting that $d\boldsymbol{r}/dt = \boldsymbol{v}$, we obtain its acceleration: $D\boldsymbol{v}/Dt \equiv \partial\boldsymbol{v}/\partial t + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}$. The material derivative $D\boldsymbol{v}/Dt$ differs from the partial derivative by the quadratically nonlinear 'advection' term $(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}$. By Newton's 2^{nd} law, the force acting on the element must equal $\rho\, \delta V\, \frac{D\boldsymbol{v}}{Dt}$.

We consider two sorts of forces acting on the fluid element. There can be an external force field such as gravity (called a body force) acting on the fluid. It may be expressed as $\boldsymbol{f}\delta V$ where $\boldsymbol{f}(\boldsymbol{r})$ is the body force per unit volume (e.g., $\boldsymbol{f} = \rho\boldsymbol{g}$ where $\boldsymbol{g}$ is the acceleration due to gravity). In addition, we have the surface force due to the pressure exerted on the element by the fluid surrounding the element. To calculate this, assume the fluid element is a cuboid with sides dx, dy, dz. The net pressure force in the $\hat{x}$ direction is the product of the area $dy\, dz$ and pressure differential between the two faces: $\delta F_x = -\frac{\partial p}{\partial x} dx \times dy\, dz$. The $-$ sign arises because if p is greater on the right face of the element compared to the left face, then the net force would be leftward. Thus, the total surface force[19] on the fluid element is $\delta\boldsymbol{F} = -(\boldsymbol{\nabla} p)\delta V$.

[19] More generally, the force due to pressure across the surface $\partial(\delta V)$ of the element is

$$\delta\boldsymbol{F}_{\text{surface}} = -\int_{\partial(\delta V)} p\hat{\boldsymbol{n}}\, dS = -\int_{\delta V} \boldsymbol{\nabla} p\, dV \approx -\boldsymbol{\nabla} p\, \delta V. \tag{19}$$

Thus, Newton's 2nd law for the fluid element reads

$$\rho\,\delta V\,\frac{D\boldsymbol{v}}{Dt} = -(\boldsymbol{\nabla} p)\,\delta V + \boldsymbol{f}\,\delta V. \tag{20}$$

Dividing by δV, we get Euler's celebrated equation of motion[20] for an inviscid fluid. It must be considered in conjunction with the continuity equation (13)

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \frac{\boldsymbol{f}}{\rho} \quad \text{and} \quad \frac{D\rho}{Dt} + \rho\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0. \tag{22}$$

Notice that the Euler equation is quadratically nonlinear in $\boldsymbol{v}$ due to the $\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v}$ advection term. This makes it difficult to solve but also allows it to describe a wide variety of ideal flows.

A vector identity allows us to write the advection term in terms of the *vorticity* $\boldsymbol{w} = \boldsymbol{\nabla}\times\boldsymbol{v}$ and a gradient term:

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{w}\times\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla} p - \frac{1}{2}\boldsymbol{\nabla} v^2 + \frac{\boldsymbol{f}}{\rho} \tag{23}$$

Here $\boldsymbol{w}\times\boldsymbol{v}$ is called the vortex force per unit mass or Lamb vector. We will have more to say about vorticity in Sect. 13.

The Euler and continuity equations are first order in time derivatives of $\boldsymbol{v}$ and ρ. So we need to specify the initial values $\rho(\boldsymbol{r},0)$ and $\boldsymbol{v}(\boldsymbol{r},0)$, to be able to evolve them forward in time[21]. However, these are still only four evolution equations for five unknown functions (density, pressure and three components of the velocity field). In particular, we have not specified how the pressure evolves in time. We will address this question for adiabatic flow in Sect. 10. Here, we deal with the slightly simpler case of incompressible constant density flow.

Pressure for constant density flow. If $\rho(\boldsymbol{r},t) = \bar\rho$ is a constant in space and time, then the continuity equation (14) implies $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$. Taking the divergence of the Euler equation (22) (in the absence of external body forces), the time derivative term is eliminated leaving us with a nondynamical 'constraint' equation

$$\boldsymbol{\nabla}^2 p = -\bar\rho\,\boldsymbol{\nabla}\cdot(\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v}). \tag{24}$$

We have used a corollary of Gauss' divergence theorem to convert the surface integral to a volume integral and taken $\boldsymbol{\nabla} p$ to be constant over the small volume δV. The minus sign is because $\hat{\boldsymbol{n}}$ is the outward-pointing normal.

[20] The Euler equation can be written in terms of the stress tensor $T_{ij} = p\delta_{ij}$

$$\partial_t v_i + v_j\partial_j v_i = -\frac{1}{\rho}\partial_j T_{ij} + \frac{1}{\rho} f_i \quad \text{in Cartesian components.} \tag{21}$$

The equation may be generalized to viscous flows by including tangential stresses in T_{ij} (see Sect. 19). Here, repeated indices are summed and no distinction is made between upper and lower indices.

[21] In addition, we need to impose suitable *boundary conditions*. The Euler and continuity equations are first order in space derivatives, and we may impose conditions on the boundary values of $\boldsymbol{v}$ and ρ. On fixed impenetrable boundaries, the normal component $\boldsymbol{v}\cdot\hat{\boldsymbol{n}}$ must vanish. In the absence of viscosity, the tangential component of $\boldsymbol{v}$ is unconstrained on boundaries. In unbounded regions, we typically have decaying BCs: $\boldsymbol{v}\to 0$ and $\rho\to\rho_0$ as $|\boldsymbol{r}|\to\infty$.

If we view the RHS as a source, this is Poisson's equation[22] for p. It can be solved with suitable boundary conditions, say using Green's function for the Laplace operator. For decaying BCs, we have

$$p(\boldsymbol{r}, t) = \frac{\bar{\rho}}{4\pi} \int \frac{\boldsymbol{\nabla} \cdot (\boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v})(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} d\boldsymbol{r}'. \tag{25}$$

Thus, for constant density, we have been able to eliminate the pressure from the Euler equation, which becomes an evolution equation for $\boldsymbol{v}$ alone. We say that in constant density flow, the pressure is not dynamical. It does not obey an independent evolution equation but is determined by the instantaneous velocity distribution. See Prob. **??** for the case of incompressible flow with variable density. □

10 Ideal adiabatic flow: entropy advection and equation of state

As pointed out below Eq. (22), the Euler and continuity equations (22) are generally an underdetermined system: they do not tell us how the pressure evolves. To understand how the pressure evolves, we need to broaden our physical perspective. Recall from Sect. 3 that a fluid can usually be considered to be in local thermal equilibrium. This means there is a local temperature field $T(\boldsymbol{r}, t)$ that, along with the pressure and density, satisfies an equation of state ($p = \rho k_b T/m$ for an ideal gas with molecular mass m.). To find the remaining dynamical equation, it is fruitful to ask how the conjugate variable to T, i.e., the entropy evolves. For a dissipationless flow, it is physically reasonable to suppose that the entropy of a fluid element remains constant in time, just as its mass does. In other words, there is no entropy production or heat exchanged between fluid elements. Such a flow is called adiabatic.

Dynamics of specific entropy. Now consider a small fluid element of volume δV and let s denote the specific entropy field (entropy per unit mass). Then the entropy of the fluid element is $\rho\, s\, \delta V$. If this is conserved as the element moves around, then its material derivative must vanish.

Using the Leibniz rule and (10), we get

$$\frac{D(\rho s \delta V)}{Dt} = \rho s (\boldsymbol{\nabla} \cdot \boldsymbol{v}) \delta V + \delta V \frac{D(\rho s)}{Dt} = 0 \quad \text{or} \quad \partial_t(\rho s) + \boldsymbol{\nabla} \cdot (\rho s \boldsymbol{v}) = 0. \tag{26}$$

In other words, the entropy per unit volume ρs is locally conserved[23] with the corresponding entropy current given by $\rho s \boldsymbol{v}$. Using the continuity equation (14), the adiabaticity of the flow implies that s is advected by $\boldsymbol{v}$:

$$\partial_t s + \boldsymbol{v} \cdot \boldsymbol{\nabla} s = 0. \tag{27}$$

[22] In electrostatics, when the electric field is expressed in terms of an electrostatic potential ($\boldsymbol{E} = -\boldsymbol{\nabla}\phi$), Gauss' law $\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho/\epsilon_0$ leads to Poisson's equation $\boldsymbol{\nabla}^2 \phi = -\rho/\epsilon_0$, where $\rho(\boldsymbol{r})$ is the electric charge density. The solution involves the Coulomb potential, which is essentially the Green function of the Laplace operator: $\phi = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} d\boldsymbol{r}'$.

[23] Integrating over the flow domain and assuming the entropy flux across the boundary vanishes, we arrive at the global conservation of entropy $\frac{d}{dt} \int \rho\, s\, d\boldsymbol{r} = 0$.

This is our third evolution equation. The pressure is then determined by the equation of state, which may be regarded as a relation among s, p and ρ. For instance, for an ideal gas with constant specific heat ratio $\gamma = c_p/c_v$, the equation of state is

$$s = c_v \log\left(\frac{p/\bar{p}}{(\rho/\bar{\rho})^\gamma}\right) \tag{28}$$

for some reference values $\bar{p}$ and $\bar{\rho}$ (see Prob. **??**).

Internal energy or pressure equation. We may also combine this equation of state (28), the entropy advection equation (27) and the continuity equation (14) to derive an evolution equation for pressure (see Prob. **??**):

$$\left(\frac{p}{\gamma-1}\right)_t + p\boldsymbol{\nabla}\cdot\boldsymbol{v} + \boldsymbol{\nabla}\cdot\left(\frac{p\boldsymbol{v}}{\gamma-1}\right) = 0. \tag{29}$$

This is called the internal energy equation since $p/(\gamma-1)$ will be interpreted as the internal energy density of an ideal gas (see Sect. 15).

Homentropic and barotropic flow. Homentropic flow is a situation where the entropy advection equation (27) can be eliminated. Here, the specific entropy $s = s_0$ is independent of both space and time and (27) is identically satisfied. Moreover, the equation of state then becomes a relation between ρ and p. In general, a relation between ρ and p is called a barotropic relation. For example[24], for homentropic flow of an ideal gas with adiabatic index γ, the barotropic relation can be written as $(p/p_0) = (\rho/\rho_0)^\gamma$ for some reference values p_0 and ρ_0. For barotropic flow, pressure $p(\boldsymbol{r},t)$ is determined by the instantaneous density $\rho(\boldsymbol{r},t)$ and we do not need to supplement the continuity and Euler equations by a third evolution equation.

Remark. Note that for $\gamma = 1$, the barotropic relation for homentropic flow of an ideal gas becomes $p = (p_0/\rho_0)\rho$ where p_0/ρ_0 is a constant. Comparing with the ideal gas law $p = (k_b T/m)\rho$, we infer that the temperature in such a flow, $T = mp_0/k_b\rho_0$, is spatially constant and independent of time. Thus, such a flow must be isothermal. However, not all isothermal flows arise this way. A gas with $\gamma = c_p/c_v \neq 1$ can display an isothermal flow. □

An important consequence of a barotropic relation expressing $\rho = \rho(p)$ is that the pressure term on the RHS of the Euler equation (23) can be expressed as a gradient:

$$\frac{\boldsymbol{\nabla} p}{\rho} = \boldsymbol{\nabla} h \quad \text{where} \quad h(p) = \int_{p_0}^{p} \frac{dp'}{\rho(p')} \quad \text{so} \quad \boldsymbol{\nabla} h = h'(p)\boldsymbol{\nabla} p = \frac{1}{\rho}\boldsymbol{\nabla} p. \tag{30}$$

For barotropic (homentropic) flow of an ideal gas,

$$\frac{\boldsymbol{\nabla} p}{\rho} = \gamma\frac{p_0}{\rho_0}\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}\frac{\boldsymbol{\nabla}\rho}{\rho} = \frac{\gamma}{\gamma-1}\boldsymbol{\nabla}\left(\frac{p}{\rho}\right) \quad \Rightarrow \quad h = \frac{\gamma}{\gamma-1}\left(\frac{p}{\rho}\right). \tag{31}$$

[24] Another example of barotropic flow is the isothermal inviscid compressible flow of an ideal gas. The barotropic relation is $p = \rho k_b T/m$ where T is the constant temperature and m the mass of a molecule. In this case, the role of specific enthalpy is played by the specific Gibbs free energy $g(\rho) = (k_b T/m)\log(\rho/\rho_0)$ which is determined up to a constant by $\boldsymbol{\nabla} g = (\boldsymbol{\nabla} p)/\rho$.

Here, $h(\rho)$ is called the specific enthalpy or enthalpy per unit mass[25]. If, in addition, the body force per unit mass can be expressed as a gradient, $\boldsymbol{f}/\rho = -\boldsymbol{\nabla}\varphi$ [i.e., body force is conservative], then the RHS of Euler's equation (22) becomes a gradient:

$$\frac{D\boldsymbol{v}}{Dt} = \partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = -\boldsymbol{\nabla}(h+\varphi). \tag{32}$$

What is more, using the identity (**??**) to write the advection term in terms of the vortex force, the Euler equation becomes

$$\partial_t \boldsymbol{v} + \boldsymbol{w}\times\boldsymbol{v} = -\boldsymbol{\nabla}(\sigma+\varphi) \quad \text{where} \quad \sigma = h + \frac{1}{2}v^2. \tag{33}$$

Here, σ is called the *stagnation enthalpy*, it reduces to the enthalpy at a stagnation point (i.e., one where $\boldsymbol{v} = 0$).

11 Bernoulli's equation

Bernoulli's principle for steady flow. Recall from Sect. 5, that a fluid flow is steady if $\boldsymbol{v}, \rho, p$, etc., are not explicitly dependent on time. In its simplest form, Bernoulli's principle concerns a drop in pressure along a streamline in places where a steady constant density flow speeds up. Euler's equation (33) for a steady homentropic flow with specific enthalpy $h(\rho)$ and body force potential φ is

$$\boldsymbol{v}\times\boldsymbol{w} = \boldsymbol{\nabla}\left(\frac{1}{2}\boldsymbol{v}^2 + h + \varphi\right) \quad \text{where} \quad \boldsymbol{w} = \boldsymbol{\nabla}\times\boldsymbol{v}. \tag{34}$$

For example, $\varphi = gz$ for the gravitational body force, where z is the vertical height and g the magnitude of the acceleration due to gravity. The left member is orthogonal to $\boldsymbol{v}$, so upon taking the dot product with the velocity field, we get Bernoulli's equation:

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}\mathcal{B} = 0 \quad \text{where} \quad \mathcal{B} = \frac{1}{2}\boldsymbol{v}^2 + h + \varphi. \tag{35}$$

Thus, the component of the gradient of the Bernoulli specific energy $\mathcal{B}$ along the velocity vector field is zero. If $\boldsymbol{r}(s)$ is a streamline[26], then Bernoulli's equation becomes

$$\frac{d\boldsymbol{r}}{ds}\cdot\boldsymbol{\nabla}\mathcal{B} = 0 \quad \text{or} \quad \frac{d\mathcal{B}(\boldsymbol{r}(s))}{ds} = 0. \tag{36}$$

So in steady flow, $\mathcal{B} = \frac{1}{2}\boldsymbol{v}^2 + h + \varphi$ is constant along streamlines. Note that $\mathcal{B}$ will, in general, take different values for different streamlines. Now recall that the enthalpy per unit mass is $h = \varepsilon + \frac{p}{\rho}$ where ε is the internal energy per unit mass, p the pressure

[25] The first law of thermodynamics $dU = TdS - pdV$, when written in terms of enthalpy $H = U + pV$ instead of internal energy U, becomes $dH = TdS + Vdp$. For an isentropic process $dS = 0$, so $dh = dp/\rho$. Here $V = M/\rho$ is the volume, M the mass of fluid, $h = H/M$ the enthalpy per unit mass, T absolute temperature and S the entropy.

[26] A streamline $\boldsymbol{r}(s)$ is an integral curve of the velocity vector field: $\frac{d\boldsymbol{r}}{ds} = \boldsymbol{v}(\boldsymbol{r}(s))$. Here, s is a parameter along the streamline.

and ρ the density. Thus, for steady homentropic inviscid flow subject to a conservative body force, Bernoulli's equation says that along streamlines,

$$\mathcal{B} = \frac{1}{2}v^2 + \varepsilon + \frac{p}{\rho} + \varphi \quad \text{is conserved.} \tag{37}$$

If the flow is incompressible, then ρ is constant along the flow and, in particular, along streamlines of the steady flow. Suppose the internal energy density of the fluid is also constant along the flow. Then we find that $\frac{1}{2}\rho v^2 + p + \rho\varphi$ is constant along streamlines. If in addition, the body force potential φ does not vary along the streamline (as for horizontal streamlines in a vertical gravitational field), then $\frac{1}{2}\rho v^2 + p$ is constant along streamlines. In other words, in regions of high pressure along a streamline, the fluid speed must be low and vice-versa. Such a situation is approximately encountered in laminar flow through a cylindrical pipe of varying cross section. On account of mass conservation, the water speeds up in regions where there is a constriction in the pipe. At such constrictions, the pressure drops, as can be demonstrated by comparing the pressure with atmospheric pressure (a lower pressure supports a shorter vertical column of water against atmospheric pressure).

Bernoulli equation for unsteady flow. There is a version of the Bernoulli equation (35) that applies to unsteady flows, though in the restricted context of barotropic potential flow ($\boldsymbol{v} = \boldsymbol{\nabla}\phi$). Potential flow is irrotational $\boldsymbol{w} = \boldsymbol{\nabla} \times \boldsymbol{v} = 0$, so the vortex force vanishes and the Euler equation (33) for barotropic flow subject to a body force derived from a potential ($\boldsymbol{f}/\rho = -\boldsymbol{\nabla}\varphi$) becomes

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{\nabla} h - \boldsymbol{\nabla}\left(\frac{1}{2}v^2\right) - \boldsymbol{\nabla}\varphi \quad \text{or} \quad \boldsymbol{\nabla}\left(h + \frac{\partial \phi}{\partial t} + \frac{1}{2}v^2 + \varphi\right) = 0. \tag{38}$$

The quantity in parentheses must be independent of location but could depend on time. Thus, we arrive at the unsteady Bernoulli equation for barotropic potential flow:

$$\frac{\partial \phi}{\partial t} + h + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \varphi = B(t). \tag{39}$$

The simplest case is that of constant density, were $h = p/\rho$. Unlike Bernoulli's equation (35) for steady flow, (39) holds throughout the fluid and is not associated with streamlines. The unsteady Bernoulli equation may also be interpreted as an evolution equation for the velocity potential ϕ. It can also be used to eliminate the pressure p in favor of the velocity potential when computing the force due to pressure[27] on a body immersed in a fluid.

12 Sound waves in homentropic flow

By sound waves, we usually mean small oscillations of the density, pressure and velocity fields around a 'background' flow. They arise in compressible flows, where

[27] If S is a surface with fluid to one side of it, then the force on the surface due to fluid pressure is given by $\int_S p\,\hat{\boldsymbol{n}}\,dA$ where dA is the area element and $\hat{\boldsymbol{n}}$ is the unit normal pointing away from the fluid.

regions of compression and rarefaction can form and propagate. It is these pressure variations that our ears receive and help us perceive as sound. For simplicity, we shall consider sound waves in the background of a motionless homogeneous fluid.

Notice first that in the absence of body forces, a fluid at rest ($\boldsymbol{v} = 0$) with *constant* pressure and density ($p = p_0$, $\rho = \rho_0$) is a static solution to the continuity and Euler equations (22):

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0 \quad \text{and} \quad \rho(\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}) = -\boldsymbol{\nabla} p. \tag{40}$$

Now suppose the still fluid suffers a disturbance resulting in small variations

$$\boldsymbol{v} = 0 + \boldsymbol{v}_1(\boldsymbol{r}, t), \quad p = p_0 + p_1(\boldsymbol{r}, t) \quad \text{and} \quad \rho = \rho_0 + \rho_1(\boldsymbol{r}, t). \tag{41}$$

The perturbations $\boldsymbol{v}_1(\boldsymbol{r}, t), p_1(\boldsymbol{r}, t)$ and $\rho_1(\boldsymbol{r}, t)$ must be such that $\boldsymbol{v}, p$ and ρ satisfy the continuity and Euler equations with $\boldsymbol{v}_1, p_1, \rho_1$ treated to linear order. However, we need to supplement (40) with another equation, as we currently have only four equations for five unknowns: ρ, p and the 3 components of $\boldsymbol{v}$. Following Laplace, we will consider the physically realistic case of sound waves in adiabatic flow of an ideal gas[28]. As discussed in (27), specific entropy is advected by adiabatic flow: $\partial_t s + \boldsymbol{v} \cdot \boldsymbol{\nabla} s = 0$. A steady uniform specific entropy $s = s_0$ is clearly a valid background solution. Putting $s = s_0 + s_1(\boldsymbol{r}, t)$ and linearizing around the static homogeneous background, we get

$$\partial_t s_1 + \boldsymbol{v}_1 \cdot \boldsymbol{\nabla} s_1 \approx \partial_t s_1 = 0. \tag{42}$$

The simplest solution is the one with vanishing entropy perturbation $s_1(\boldsymbol{r}, t) \equiv 0$, corresponding to homentropic flow $s(\boldsymbol{r}, t) \equiv s_0$. In particular, the specific entropy of every fluid element is the same and remains that way at all times. Now, for homentropic flow, the thermodynamic equation of state relating p, ρ and s reduces to a barotropic relation between pressure and density $p = p(\rho, s_0)$ [such as $p/p_0 = (\rho/\rho_0)^\gamma$ for adiabatic flow of an ideal gas]. Inserting (41) in the barotropic relation and using the leading Taylor approximation, we get

$$p = p_0 + \left(\frac{\partial p}{\partial \rho}\right)_{\rho_0, s_0} \rho_1 + \cdots \quad \Rightarrow \quad p_1 \approx \left(\frac{\partial p}{\partial \rho}\right)_{\rho_0, s_0} \rho_1. \tag{43}$$

Thus, the small pressure and density variations are proportional. The constant of proportionality[29] is denoted (celeritas means velocity in Latin)

$$c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_{\rho_0, s_0} = \frac{K_s}{\rho_0} \tag{44}$$

and has dimensions of the square of a speed. We will show that c_s (called the adiabatic sound speed) is the speed at which sound waves propagate. To do so, we derive an

[28] In his Principia, Newton computed the speed of sound in air assuming the flow to be isothermal. This was found not to be a particularly good approximation, since there are temperature fluctuations in a sound wave. Laplace's assumption of adiabatic flow led to a value closer to experimental measurements.

[29] Here, K_s is the isentropic bulk modulus (see Sect. 7), a measure of stiffness of the medium.

equation for sound waves by linearizing the continuity and Euler equations around a motionless fluid. Ignoring products of small quantities $\boldsymbol{v}_1, p_1$ and ρ_1, the continuity and Euler equations

$$\begin{aligned} \partial_t(\rho_0+\rho_1)+\boldsymbol{\nabla}\cdot((\rho_0+\rho_1)\boldsymbol{v}_1) &= 0 \quad \text{and} \\ (\rho_0+\rho_1)(\partial_t \boldsymbol{v}_1+\boldsymbol{v}_1\cdot\boldsymbol{\nabla}\boldsymbol{v}_1) &= -\boldsymbol{\nabla}(p_0+p_1) \end{aligned} \tag{45}$$

become

$$\partial_t\rho_1+\rho_0\boldsymbol{\nabla}\cdot\boldsymbol{v}_1=0 \quad \text{and} \quad \rho_0\partial_t\boldsymbol{v}_1=-\boldsymbol{\nabla}p_1. \tag{46}$$

Putting $p_1 = c_s^2\rho_1$ and taking a divergence, the linearized Euler equation becomes

$$\rho_0\partial_t(\boldsymbol{\nabla}\cdot\boldsymbol{v}_1) = -c_s^2\boldsymbol{\nabla}^2\rho_1. \tag{47}$$

Eliminating $\boldsymbol{\nabla}\cdot\boldsymbol{v}_1 = -\rho_0^{-1}\partial_t\rho_1$ using the continuity equation, we get

$$\partial_t^2\rho_1 = c_s^2\boldsymbol{\nabla}^2\rho_1, \tag{48}$$

which we recognize as the 3d d'Alembert wave equation for density variations. We deduce that c_s may be interpreted as the speed at which sound waves propagate. The corresponding equations for pressure and velocity perturbations are the subject of Prob. **??**.

As we might physically expect from Sect. 7, for incompressible flow ($\rho=\rho_0, \rho_1 = 0$), the sound speed $c_s^2 = \frac{p_1}{\rho_1} = \frac{\delta p}{\delta\rho} \to \infty$ as the density variations are vanishingly small even for large pressure variations. Thus, sound waves travel much faster than the fluid in the incompressible limit and the Mach number $M = |\boldsymbol{v}|/c_s$ tends to zero.

13 Vorticity and its evolution

Vorticity $\boldsymbol{w} = \boldsymbol{\nabla}\times\boldsymbol{v}$ is the curl of the velocity field[30]. Unlike $\boldsymbol{v}$, which is a polar vector (reverses sign under the reflection $\boldsymbol{r}\to-\boldsymbol{r}$), $\boldsymbol{w}$ is a pseudovector or axial vector (no change in sign under reflections). Since the divergence of a curl vanishes, vorticity is solenoidal: $\boldsymbol{\nabla}\cdot\boldsymbol{w} = 0$. Vorticity has dimensions of (1/time) and is a measure of local rotation in a flow. A flow without vorticity is called irrotational[31]. For example, a bucket of fluid rigidly rotating at small angular velocity $\boldsymbol{\Omega} = \Omega\hat{z}$ has the azimuthal velocity field $\boldsymbol{v}(r,\theta,z) = \boldsymbol{\Omega}\times\boldsymbol{r} = \Omega(x\hat{y}-y\hat{x}) = \Omega r\hat{\theta}$ in cylindrical coordinates (see Fig. 6a). The corresponding vorticity $\boldsymbol{w} = \boldsymbol{\nabla}\times\boldsymbol{v} = \frac{1}{r}\partial_r(rv_\theta)\hat{z}$ is vertically upwards and constant over the bucket. In fact, as pointed out by Stokes, $\boldsymbol{w} = \Omega(\partial_x x+\partial_y y)\hat{z} = 2\Omega\hat{z}$ has a magnitude of twice the angular speed Ω. Eddies or vortices are manifestations of vorticity in a flow. They are ubiquitous[32] in flows and can come in various sizes: in a wash basin, in the sea and in the atmosphere.

[30] In general, the vorticity field need not be orthogonal to velocity. The flow helicity density $\mathscr{K} = \boldsymbol{v}\cdot\boldsymbol{w}$ is a local measure of the extent to which $\boldsymbol{v}$ and $\boldsymbol{w}$ fail to be orthogonal (see Prob. **??**). Flow helicity density is a pseudoscalar, it changes sign under reflections.

[31] Potential flow, i.e., where $\boldsymbol{v} = \boldsymbol{\nabla}\phi$ is the gradient of a scalar velocity potential is irrotational. Locally, an irrotational velocity field must be a gradient.

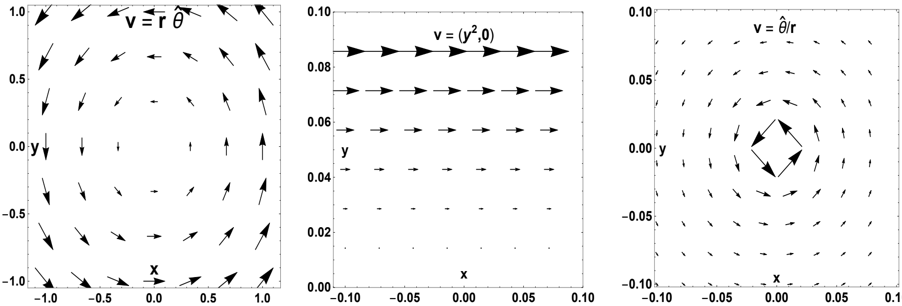
[32] There are many names for vortex-like structures: swirls, whirlpools, whorls, cyclones, hurricanes, typhoons, tornadoes, maelstroms, etc. Leonardo da Vinci was fascinated by vortices. Many of his sketches contain detailed illustrations of eddies in fluids. He even noticed similarities between the train of vortices in the wake behind a flat plate and braided hair!

Given a closed contour C in a fluid, the *circulation* $\Gamma(C) = \oint_C \boldsymbol{v} \cdot d\boldsymbol{l}$ measures how much $\boldsymbol{v}$ 'goes round' C. For a simple closed contour C, Stokes' theorem implies that the circulation is the flux of vorticity across any surface S in the fluid that spans C (i.e., whose boundary ∂S is C):

$$\Gamma(C) = \oint_C \boldsymbol{v} \cdot d\boldsymbol{l} = \int_S (\boldsymbol{\nabla} \times \boldsymbol{v}) \cdot d\boldsymbol{S} = \int_S \boldsymbol{w} \cdot d\boldsymbol{S} \quad \text{where} \quad \partial S = C. \tag{49}$$

Enstrophy. The square of the L^2 norm of vorticity $\int \boldsymbol{w}^2 \, d\boldsymbol{r}$ is called enstrophy. It is a global measure of vorticity. We will see that it is conserved in incompressible barotropic 2d flows, but not in 3d, where it can grow due to 'vortex stretching'.

Example: shear flow. Shears are large scissors used, for instance, to trim the wool of sheep. The blades of the scissors are said to slide over each other in a 'shearing' motion. By analogy, a flow is a shear flow if layers if fluid slide over each other at different speeds. The shear flow $\boldsymbol{v}(x, y, z) = (u(y), 0, 0)$ with horizontal streamlines illustrated in Fig. 6b is an example of a flow with vorticity. Here, horizontal layers of fluid move at different speeds depending on their height y, leading to the vorticity $\boldsymbol{w} = \boldsymbol{\nabla} \times \boldsymbol{v} = -u'(y)\hat{z}$. When there is a velocity differential between two layers of fluid, one can imagine that a little windmill placed there would start spinning. In fact, eddies can be produced as the interface 'curls up'. □



(a) (b) (c)

Figure 6: Velocity vector field for (a) rigidly rotating bucket of fluid $\boldsymbol{v} = \Omega r \hat{\theta}$ for $\Omega = 1$, (b) shear flow $\boldsymbol{v}(x, y) = (u(y), 0)$ for $u(y) = y^2$ and (c) point-like vortex $\boldsymbol{v} = (\alpha/r)\hat{\theta}$ for $\alpha = 1$.

Example: point-like vortex. For $\alpha > 0$, the planar azimuthal velocity field $\boldsymbol{v}(r, \theta) = \frac{\alpha}{r}\hat{\theta}$ shown in Fig. 6c has counterclockwise circular streamlines. It has no vorticity $\boldsymbol{w} = \frac{1}{r}\partial_r(r\frac{\alpha}{r})\hat{z} = 0$ except at $r = 0$: $\boldsymbol{w} = 2\pi\alpha\delta^2(\boldsymbol{r})\hat{z}$. Thus, the vorticity is concentrated at the origin. The constant $2\pi\alpha$ comes from requiring the flux of $\boldsymbol{w}$ to equal the circulation of $\boldsymbol{v}$ around any contour enclosing the origin

$$\oint \boldsymbol{v} \cdot d\boldsymbol{l} = \oint (\alpha/r) r \, d\theta = 2\pi\alpha. \tag{50}$$

If we include the vertical z direction, this point vortex becomes a 'line' vortex, with vorticity concentrated along the z axis.

More generally, vortices can take the shape of tubes and rings (see Sect. 14). Smoke rings are examples of vortex tubes. Dolphins blow vortex rings in water and chase them. Kelvin and Helmholtz discovered many interesting properties of vortex tubes. Inviscid fluid flow tends to stretch and bend vortex tubes while carrying them along. They survive in the absence of viscosity but dissipate due to friction (see Sect. 19). □

Vorticity evolution. Let us obtain the equation for vorticity by taking the curl of the Euler equation. It is simplest to do this for barotropic (e.g., homentropic) flow of a fluid subject to a conservative body force. Thus, we suppose that $\boldsymbol{\nabla} p/\rho = \boldsymbol{\nabla} h$ and that the body force per unit mass is $\boldsymbol{f}/\rho = -\boldsymbol{\nabla}\varphi$, so that (33) becomes

$$\partial_t \boldsymbol{v} + \boldsymbol{w} \times \boldsymbol{v} = -\boldsymbol{\nabla}\left(h + \varphi + (1/2)\boldsymbol{v}^2\right). \tag{51}$$

Taking a curl, the RHS vanishes and we get the vorticity evolution equation

$$\frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{\nabla} \times (\boldsymbol{w} \times \boldsymbol{v}) = 0. \tag{52}$$

In this form, both the pressure and density have been eliminated from the Euler equation! This comes at the cost of making it second order in spatial derivatives of $\boldsymbol{v}$. Using $\boldsymbol{\nabla} \cdot \boldsymbol{w} = 0$ and the vector identity

$$\boldsymbol{\nabla} \times (\boldsymbol{w} \times \boldsymbol{v}) = ((\boldsymbol{\nabla} \cdot \boldsymbol{v}) + \boldsymbol{v} \cdot \boldsymbol{\nabla})\, \boldsymbol{w} - ((\boldsymbol{\nabla} \cdot \boldsymbol{w}) + \boldsymbol{w} \cdot \boldsymbol{\nabla})\, \boldsymbol{v}, \tag{53}$$

the evolution equation for vorticity becomes

$$\frac{D\boldsymbol{w}}{Dt} - (\boldsymbol{w} \cdot \boldsymbol{\nabla})\boldsymbol{v} + (\boldsymbol{\nabla} \cdot \boldsymbol{v})\boldsymbol{w} = 0. \tag{54}$$

Notice that if the flow is incompressible ($\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$), the last term vanishes and we get $\frac{D\boldsymbol{w}}{Dt} = (\boldsymbol{w} \cdot \boldsymbol{\nabla})\boldsymbol{v}$. One consequence of this equation is that vorticity is *frozen* into an inviscid barotropic flow, as we will see in Sect. 14. Heuristically, what this means is that vortices (more precisely vortex tubes or rings) are dragged along by the flow field $\boldsymbol{v}$.

Advection of vorticity in 2d incompressible barotropic flow. Vorticity behaves in a particularly simple manner in 2d incompressible barotropic flows. Consider flow on a portion of the x-y plane, so that $\boldsymbol{v} = (u, v, 0)$ while the vorticity points vertically $\boldsymbol{w} = \boldsymbol{\nabla} \times \boldsymbol{v} = w\,\hat{z}$ where[33] $w = v_x - u_y$. It follows that $(\boldsymbol{w} \cdot \boldsymbol{\nabla})\boldsymbol{v} = 0$ and (54) becomes $\frac{D\boldsymbol{w}}{Dt} = 0$. Thus, vorticity is advected by a planar incompressible barotropic flow. In particular, the vorticity of a fluid element is constant as the element moves around. Using incompressibility ($\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$), this can be expressed as a local conservation law for w:

$$\frac{D\boldsymbol{w}}{Dt} = 0 \quad \Rightarrow \quad \frac{\partial w}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} w = 0 \quad \Rightarrow \quad \frac{\partial w}{\partial t} + \boldsymbol{\nabla} \cdot (w\boldsymbol{v}) = 0. \tag{55}$$

[33] The (negative) Laplacian of the stream function ψ (11) is the vorticity component w in 2d incompressible flow. In fact, $w = v_x - u_y = -(\psi_{xx} + \psi_{yy}) = -\boldsymbol{\nabla}^2\psi$.

Multiplying by w, we get a local conservation law for w^2 as well:

$$\frac{1}{2}\frac{\partial w^2}{\partial t} + \boldsymbol{\nabla} \cdot \left(\frac{1}{2} w^2 \boldsymbol{v}\right) = 0. \tag{56}$$

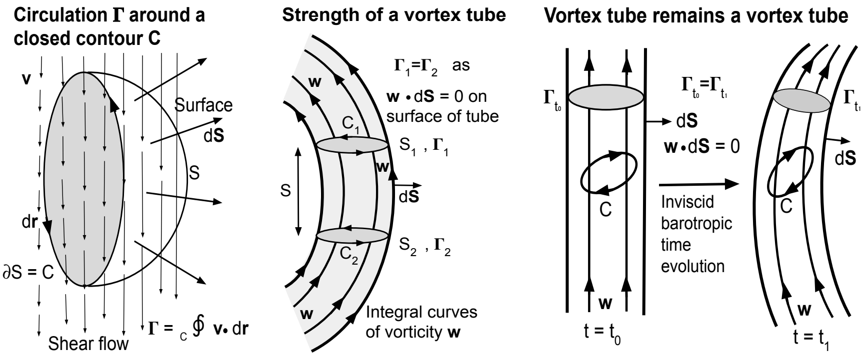
In fact, proceeding this way, one may show (see Prob. **??**) that w^n is locally conserved with the current $w^n \boldsymbol{v}$ for $n = 1, 2, 3, \ldots$. Integrating over the plane and assuming $\boldsymbol{v}$ vanishes on boundaries and sufficiently fast at infinity, we get conservation laws for the moments of vorticity

$$\frac{d}{dt} \int w^n \, dx \, dy = 0 \quad \text{for} \quad n = 1, 2, 3, \ldots. \tag{57}$$

Enstrophy is defined as the integral of the square of vorticity. Thus, enstrophy and its higher cousins[34] are invariants for 2d incompressible barotropic flows.

14 Vortex tubes: Kelvin and Helmholtz theorems

In three dimensions, vorticity is frozen (54) into inviscid barotropic flow. One manifestation of this is Kelvin's theorem on conservation of circulation.



(a) (b) (c)

Figure 7: (a) Circulation around a closed contour. (b) Integral curves of vorticity. Strength of a vortex tube is independent of choice of encircling closed contour. C_1 and C_2 wind once around the tube, they are noncontractible. (c) A vortex tube remains a vortex tube under inviscid barotropic flow. The curve C is a contractible closed curve lying on the surface of the tube, it does not wind around the tube. $d\boldsymbol{S}$ is an area element on the surface of the vortex tube.

Vorticity is the curl of the velocity field just as the magnetic field is the curl of a vector potential $\boldsymbol{A}$. Like the magnetic flux across an oriented surface S, we may consider the flux Γ of vorticity across a surface S in a fluid. By Stokes' theorem,

$$\Gamma = \int_S \boldsymbol{w} \cdot d\boldsymbol{S} = \int_S (\boldsymbol{\nabla} \times \boldsymbol{v}) \cdot d\boldsymbol{s} = \oint_C \boldsymbol{v} \cdot d\boldsymbol{r} \tag{58}$$

[34] In fact, subject to suitable behavior at the boundary, the integral of any function of vorticity (that can be approximated by polynomials) is conserved.

is the circulation (49) or line integral of $\boldsymbol{v}$ around the directed closed curve $C = \partial S$ that bounds S (see Fig. 7a).

Kelvin's law: conservation of circulation around a material loop. Suppose C_t at time t is a closed material contour. Then $\Gamma(C_t)$ is conserved as C_t is transported by an inviscid (possibly compressible) barotropic flow subject to conservative body forces. Let us show that the time derivative[35] of Γ vanishes:

$$\frac{d\Gamma}{dt} = \oint_{C_t} \frac{D\boldsymbol{v}}{Dt} \cdot d\boldsymbol{r} + \oint_{C_t} \boldsymbol{v} \cdot \frac{Dd\boldsymbol{r}}{Dt}. \tag{59}$$

Since C_t moves with the flow, we used material derivatives and the product rule that it satisfies. The velocity $\boldsymbol{v}(\boldsymbol{r}, t)$ is that of a moving material element and we must also account for the motion of the material element $d\boldsymbol{r}$ itself. For barotropic flow with conservative body forces, Euler's equation (33) says that $\frac{D\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla}(h + \varphi)$. On the other hand, the material derivative of a material line element is its fluid velocity element: $\frac{Dd\boldsymbol{r}}{Dt} = d\boldsymbol{v}$ and $\boldsymbol{v} \cdot d\boldsymbol{v} = \frac{1}{2} d(\boldsymbol{v}^2) = \frac{1}{2} \boldsymbol{\nabla} \boldsymbol{v}^2 \cdot d\boldsymbol{r}$. Thus, by Stokes' theorem,

$$\frac{d\Gamma}{dt} = \oint_{C_t} \boldsymbol{\nabla} \left(-h - \varphi + \frac{1}{2} \boldsymbol{v}^2 \right) \cdot d\boldsymbol{r} = 0 \tag{60}$$

as it is the line integral of a gradient around a closed curve. It follows that the circulation around a closed fluid contour is constant in time.

Alternatively, the flux of vorticity across any surface that moves with the fluid is constant in the absence of viscosity, provided the sum of pressure and body forces per unit mass may be expressed as a gradient. Loosely, in the absence of viscosity, eddies and vortices cannot develop in a barotropic flow that was initially irrotational.

Vortex tube. A vortex line is an integral curve of the vorticity field $\boldsymbol{w}$ at a given instant of time. Given any closed curve C in the fluid, consider the vortex lines through C. They form a surface, called a vortex tube, as shown in Fig. 7. The vorticity is everywhere tangent to a vortex tube. A vortex tube of infinitesimal cross section is called a vortex filament. We now describe **Helmholtz's theorem** on vortex tubes, which states that inviscid barotropic flow carries vortex tubes to vortex tubes of the same strength.

Strength of a vortex tube. At an initial time t_0, a vortex tube can be assigned a strength, the circulation $\Gamma(C)$ around a closed curve C that winds around the tube once. To be meaningful, this strength should be independent of the choice of closed loop C. This is indeed true. Suppose C_1 and C_2 are two curves winding around the vortex tube once each. For simplicity, we assume that they do not intersect[36]. Let S_1 and S_2 be any two surfaces with $\partial S_1 = C_1, \partial S_2 = C_2$ (see Fig. 7b). We assume

[35] $d\Gamma/dt$ refers to the rate of change in the circulation around a material contour, as it is carried around by the flow. $\Gamma(C_t)$ depends on a whole contour and is not a function of one position like the field $\boldsymbol{v}(\boldsymbol{r}, t)$. Bearing this in mind, one could equally well use the notation $D\Gamma/Dt$ for $d\Gamma/dt$.

[36] If C_2 is a closed curve that winds once around the vortex tube and intersects C_1, then we may still conclude that $\Gamma(C_1) = \Gamma(C_2)$ by choosing a third contour C_3 which does not intersect either C_1 or C_2 and use the argument that follows to show $\Gamma(C_1) = \Gamma(C_2) = \Gamma(C_3)$

that $S_{1,2}$ are chosen not to intersect. Consider the portion of the vortex tube between S_1 and S_2. Its interior is a solid cylindrical region R with closed surface $\partial R = \Sigma = S \cup S_1 \cup S_2$ where S is the tubular surface of the vortex tube. The flux of vorticity across Σ must vanish by the divergence theorem: $\int_\Sigma \boldsymbol{w} \cdot d\boldsymbol{S} = \int_R \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}) d\boldsymbol{r}$. Thus, assigning suitable orientations,

$$0 = \cancel{\int_S \boldsymbol{w} \cdot d\boldsymbol{S}} + \int_{S_1} \boldsymbol{w} \cdot d\boldsymbol{S} + \int_{S_2} \boldsymbol{w} \cdot d\boldsymbol{S} = \oint_{C_1} \boldsymbol{v} \cdot d\boldsymbol{l} - \oint_{C_2} \boldsymbol{v} \cdot d\boldsymbol{l}. \tag{61}$$

The integral over S vanishes since vorticity is tangential to a vortex tube. Thus, $\Gamma(C_1) = \Gamma(C_2)$: the circulation around a vortex tube is independent of the choice of encircling contour. As a consequence, *a vortex tube cannot abruptly end in the fluid*, it must close on itself to form a vortex ring or end on a boundary.

Vortex tubes evolve into vortex tubes. Inviscid isentropic flow takes a vortex tube to another vortex tube. This is a manifestation of the freezing-in of vorticity into the velocity field. To show this, we consider a vortex tube at t_0 and follow its surface as it moves with the flow up to a final time $t_1 > t_0$. We wish to show that the new tube is a vortex tube, i.e., that the vorticity is everywhere tangent to it. To this end, consider a contractible[37] closed curve $C(t_0)$ lying on the initial vortex tube, the flow maps it to a new contractible closed curve $C(t_1)$ lying on the final tube, as shown in Fig. 7c. By Kelvin's theorem, $\Gamma(C(t_0)) = 0 = \Gamma(C(t_1))$. Now, suppose S is the portion of the new tubular surface enclosed by $C(t_1)$, i.e., $\partial S = C(t_1)$. Then

$$0 = \Gamma(C(t_1)) = \int_S \boldsymbol{w} \cdot d\boldsymbol{S}. \tag{62}$$

By suitably repositioning and shrinking $C(t_0)$, this is true for an infinitesimal closed curve $C(t_1)$ around any point on the new tube. Thus, we conclude that $\boldsymbol{w} \cdot d\boldsymbol{S} = 0$ at every point of the new tube. In other words, the vorticity is everywhere tangent to the new tube, which must therefore be a vortex tube.

Strength of a vortex tube is time-independent. Suppose $C(t)$ is a closed material contour winding once round a vortex tube as it evolves in time. By Kelvin's theorem, strength of the vortex tube $\Gamma(C(t))$ is independent of time.

15 Local conservation laws for an inviscid flow

We have already encountered two local conservation laws: that for mass, the continuity equation $\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0$ (15) and that for entropy in ideal adiabatic flow, $\partial_t(\rho s) + \boldsymbol{\nabla} \cdot (\rho s \boldsymbol{v}) = 0$ (26). Integrating over the flow domain and assuming the corresponding fluxes across the boundary vanish, we get the conservation of the total mass and entropy of the fluid. There are four more such local (and corresponding global) conservation laws: for energy, linear momentum, angular momentum and helicity which we discuss below.

[37] 'Contractible' means the closed curve can be continuously shrunk to a point while remaining on the surface of the tube.

Conservation of linear momentum. By analogy with particle mechanics, we would expect the linear momentum of a fluid to be given by the vector $\boldsymbol{P} = \int \rho \boldsymbol{v}\, d\boldsymbol{r}$. We will show that with suitable BCs, $\boldsymbol{P}$ is constant in time and that the momentum density $\rho\boldsymbol{v}$ is locally conserved. In fact, it satisfies the momentum equation:

$$\partial_t(\rho v_i) + \partial_j \Pi_{ij} = 0 \quad \text{where} \quad \Pi_{ij} = p\delta_{ij} + \rho v_i v_j. \tag{63}$$

This equation is obtained by combining the continuity (15) and Euler (22) equations in the absence of external forces. Indeed, one finds that

$$\partial_t(\rho v_i) = -v_i \partial_j(\rho v_j) - \rho v_j \partial_j v_i - \partial_i p = -\partial_i p - \partial_j(\rho v_i v_j) = -\partial_j \Pi_{ij}. \tag{64}$$

It holds independent of how pressure evolves and is thus valid both for adiabatic as well as incompressible flow. The symmetric 2^{nd} rank momentum current tensor $\Pi_{ij} = T_{ij} + \rho v_i v_j$ is related to the stress tensor of Sect. 9. Its divergence appears in (63). Integrating over the flow domain Ω, we get the conservation of momentum:

$$\frac{dP_i}{dt} = -\int_\Omega \partial_j \Pi_{ij} d\boldsymbol{r} = -\int_{\partial\Omega} \Pi_{ij} n_j dS = 0. \tag{65}$$

Here, n_j are the components of the unit normal on the boundary surface $\partial\Omega$. We used the divergence theorem to convert the volume integral into a surface integral and assumed that the flux of momentum across the boundary vanishes (this is the case, for instance, with decaying BCs in $\Omega = \mathbb{R}^3$ or with periodic BCs in a cuboid).

Conservation of angular momentum. It is natural to define the angular momentum density (relative to $\boldsymbol{r} = 0$) as $\mathcal{L} = \boldsymbol{r} \times \rho\boldsymbol{v}$, where $\rho\boldsymbol{v}$ is the momentum density. In components, $\mathcal{L}_i = \rho\epsilon_{ijk} x_j p_k$. It satisfies the local conservation law

$$\partial_t \mathcal{L}_i + \partial_l \Lambda_{il} = 0 \quad \text{where} \quad \Lambda_{il} = \epsilon_{ijk} x_j \Pi_{kl} \tag{66}$$

is the angular momentum current tensor, which is built from the momentum tensor Π. This is checked by evaluating $\partial_t \mathcal{L}_i$ using (63).

It is valid irrespective of how pressure evolves (incompressible, barotropic, adiabatic, etc.) but assumes there are no external or viscous forces. The total angular momentum $\boldsymbol{L} = \int \mathcal{L} d\boldsymbol{r}$ is conserved with suitable BCs, such as decaying BCs in an infinite domain. Moreover, in an infinite domain, $\boldsymbol{L}$ is independent of the choice of origin since $\boldsymbol{r}$ is a dummy variable of integration. On the other hand, in axisymmetric domains such as a circular cylinder or torus, the component of angular momentum along the symmetry axis is conserved provided there is no flux of angular momentum across the boundary.

Conservation of energy. By analogy with the kinetic energy $\frac{1}{2}mv^2$ of a particle we expect a fluid flow to have a kinetic energy $\int \frac{1}{2}\rho v^2\, d\boldsymbol{r}$. In addition, we might expect a potential energy which could variously be thought of as an internal energy or compressional energy or thermal energy. From the equipartition principle in the kinetic theory of a gas at temperature T, each (translational, rotational, vibrational) degree of freedom of a molecule has an associated energy $\frac{1}{2}k_bT$, where k_b is Boltzmann's constant. So if there are N molecules of the gas with f degrees of freedom each, the internal energy is $(f/2)Nk_bT$. For example, a monatomic gas has $f = 3$ translational degrees of freedom, so that its internal energy takes the familiar form $(3/2)Nk_bT$. More generally, a gas with specific heat ratio γ has in effect $f = 2/(\gamma - 1)$ degrees of freedom[38]. Furthermore, if we trade the temperature for pressure using the ideal gas equation of state $pV = Nk_bT$ we find that the internal energy of a homogeneous ideal gas at equilibrium in a volume V is $pV/(\gamma - 1)$. A flowing fluid may be regarded as a collection of small volumes $d\boldsymbol{r}$ each in local thermal equilibrium with pressure $p(\boldsymbol{r})$. Thus, the internal or potential energy density of an ideal gas is $\mathcal{U} = p/(\gamma - 1)$. Adding the kinetic and potential energies, we guess that the total energy of a fluid composed of an ideal gas with specific heat ratio γ is

$$E = \int \left[\frac{1}{2}\rho v^2 + \frac{p}{\gamma - 1}\right] d\boldsymbol{r} = \int \mathcal{E}\, d\boldsymbol{r} \tag{67}$$

where $\mathcal{E}$ is the energy density.

To confirm our guess, we now show that $\mathcal{E}$ satisfies a local conservation law. To do so, we combine the inviscid adiabatic evolution equations for ρ, $\boldsymbol{v}$ (22) and p (29) in the absence of external forces

$$\rho_t = -\boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}), \quad \boldsymbol{v}_t = -\boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} - \frac{\boldsymbol{\nabla} p}{\rho}, \quad \frac{p_t}{\gamma - 1} = -p \boldsymbol{\nabla} \cdot \boldsymbol{v} - \frac{\boldsymbol{\nabla} \cdot (p\boldsymbol{v})}{\gamma - 1}, \tag{68}$$

to evaluate the time derivative of the internal energy density

$$\mathcal{E}_t = -\frac{v^2}{2} \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) - \rho \boldsymbol{v} \cdot (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{\nabla} p - p \boldsymbol{\nabla} \cdot \boldsymbol{v} - \frac{\boldsymbol{\nabla} \cdot (p\boldsymbol{v})}{\gamma - 1}. \tag{69}$$

Using the Leibniz rule $\boldsymbol{\nabla} \cdot (g\boldsymbol{u}) = \boldsymbol{\nabla} g \cdot \boldsymbol{u} + g \boldsymbol{\nabla} \cdot \boldsymbol{u}$ for any function g and vector field $\boldsymbol{u}$, the first 2 terms may be combined as a divergence, as can the last 3. Thus, in the absence of external forces, we get the local conservation law

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla} \cdot \left(\frac{v^2}{2} \rho \boldsymbol{v} + \frac{\gamma}{\gamma - 1} p \boldsymbol{v} \right) = 0. \tag{70}$$

The quantity in parentheses is the energy current vector. This formula holds for adiabatic flow as well as the special case of homentropic barotropic flow where p is a function of ρ. Integrating (70) over the flow domain, it follows that the total energy

[38] Consider N molecules of an ideal gas satisfying the equation of state $pV = Nk_bT$. The caloric condition from the Joule-Thomson porous plug experiment says that the internal energy of such a gas is independent of volume occupied and depends only on the temperature, $U = U(T, \cancel{V})$. Thus $dU = C_v dT$ where $C_v(T)$ is called the heat capacity at constant volume. Putting these in the first law of thermodynamics ($\delta Q = dU + pdV$), we get $\delta Q = (C_v + Nk_b)dT - Vdp$. Now, the heat capacity at constant pressure is $C_p = (\delta Q/\delta T)_p$. Thus, $C_p - C_v = Nk_b$. What is more, from equipartition, $U = (f/2)Nk_bT$ so $C_v = (f/2)Nk_b$ where f is the number of degrees of freedom (translational, rotational and possibly vibrational) of the molecule: $C_v = (3/2)Nk_b$ for monatomic, $(5/2)Nk_b$ for diatomic and $3Nk_b$ for noncollinear polyatomic (without vibrations). Thus, we find that $\gamma = C_p/C_v = 1 + Nk_b/C_v = 1 + 2/f$.

(67) is conserved if the flux of the energy current across the boundary vanishes. This happens, for instance, with decaying boundary conditions ($\boldsymbol{v} \to 0, \rho \to \rho_0$ sufficiently fast as $|\boldsymbol{r}| \to \infty$) or periodic boundary conditions in a cuboid.

Barotropic energy density. In homentropic flow, we have the barotropic relation $p = p_0(\rho/\rho_0)^\gamma$ corresponding to the specific enthalpy $h(\rho) = (\gamma/(\gamma-1))p/\rho$ (31). In this case, the potential energy density $\mathcal{U} = p/(\gamma-1)$ becomes $\mathcal{U}(\rho) = p_0(\rho/\rho_0)^\gamma/(\gamma-1)$. We notice that $\mathcal{U}(\rho)$ is an antiderivative of the specific enthalpy: $\mathcal{U}'(\rho) = h(\rho)$. This relation holds more generally for any barotropic relation between p and ρ.

Energy in incompressible limit. In the incompressible case ($\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$), we continue to have local and global conservation laws for energy, though there is no compressional potential energy. In this case, the time derivative of the kinetic energy density may be expressed as a divergence using the Euler equation $\boldsymbol{v}_t + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = -\boldsymbol{\nabla}p/\rho$ and continuity equation $\rho_t + \boldsymbol{v}\cdot\boldsymbol{\nabla}\rho = 0$:

$$\left(\frac{1}{2}\rho v^2\right)_t = -\frac{1}{2}v^2\boldsymbol{v}\cdot\boldsymbol{\nabla}\rho - \rho\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \boldsymbol{v}\cdot\boldsymbol{\nabla}p. \tag{71}$$

As before, the first two terms combine to give the divergence of $-\frac{1}{2}\rho v^2\boldsymbol{v}$ while the last one is the divergence of $-p\boldsymbol{v}$ since $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$. Thus, we obtain the local conservation law for energy in incompressible hydrodynamics:

$$\partial_t\left(\frac{1}{2}\rho v^2\right) + \boldsymbol{\nabla}\cdot\left\{\left(p + \frac{1}{2}\rho v^2\right)\boldsymbol{v}\right\}. \tag{72}$$

We may obtain this equation by letting $\gamma \to \infty$ in (70). This is reasonable, since $\gamma = \infty$ corresponds to zero internal degrees of freedom $f = 2/(\gamma-1) = 0$, which means the gas molecules have no internal/random thermal/compressional energy; all the energy comes from the large-scale motion of the gas via the velocity field.

Flow helicity and its conservation. Flow helicity is defined as $\mathcal{K} = \int \boldsymbol{v}\cdot\boldsymbol{w}\, d\boldsymbol{r}$. Helicity density $\mathscr{K} = \boldsymbol{v}\cdot\boldsymbol{w}$ is proportional to the component of vorticity in the direction of velocity[39]. Though $\boldsymbol{w} = \boldsymbol{\nabla}\times\boldsymbol{v}$, it is not necessarily orthogonal to $\boldsymbol{v}$. A fluid flow has helicity, for instance, if streamlines are shaped like helices. The examples of rotational planar flows in Sect. 13 have zero helicity. However, it is not difficult to come up with helical flows. The toy example $\boldsymbol{v} = (z, x, y)$ is one with nonvanishing $\mathscr{K} = \boldsymbol{v}\cdot\boldsymbol{w}$. We will show that helicity is an integral invariant for inviscid homentropic (barotropic) flow subject to conservative body forces. In fact, $\boldsymbol{v}\cdot\boldsymbol{w}$ satisfies a local conservation law:

$$\frac{\partial(\boldsymbol{v}\cdot\boldsymbol{w})}{\partial t} + \boldsymbol{\nabla}\cdot((\sigma+\varphi)\boldsymbol{w} + (\boldsymbol{w}\times\boldsymbol{v})\times\boldsymbol{v}) = 0. \tag{73}$$

To show this, we note that the Euler equation (33) $\boldsymbol{v}_t + \boldsymbol{w}\times\boldsymbol{v} = -\boldsymbol{\nabla}(\sigma+\varphi)$ and the vorticity evolution equation (52) give

$$\boldsymbol{w}\cdot\boldsymbol{v}_t = -\boldsymbol{w}\cdot\boldsymbol{\nabla}(\sigma+\varphi) \quad \text{and} \quad \boldsymbol{v}\cdot\boldsymbol{w}_t = -\boldsymbol{v}\cdot\boldsymbol{\nabla}\times(\boldsymbol{w}\times\boldsymbol{v}). \tag{74}$$

[39] The significance of helicity was pointed out relatively recently, by H K Moffat in 1969 [**?**]. A similar concept arises in particle physics, where helicity is the component of spin along the momentum of a particle.

This implies

$$\partial_t(\boldsymbol{v}\cdot\boldsymbol{w}) = -\boldsymbol{w}\cdot\boldsymbol{\nabla}(\sigma+\varphi) - \boldsymbol{v}\cdot\boldsymbol{\nabla}\times(\boldsymbol{w}\times\boldsymbol{v}). \tag{75}$$

We want to write the RHS as a divergence. The first term is a divergence since $\boldsymbol{w}$ is solenoidal: $\boldsymbol{w}\cdot\boldsymbol{\nabla}(\sigma+\varphi) = \boldsymbol{\nabla}\cdot((\sigma+\varphi)\boldsymbol{w})$. For the second term we use the vector identity ($\boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot\boldsymbol{\nabla}\times\boldsymbol{A} - \boldsymbol{A}\cdot\boldsymbol{\nabla}\times\boldsymbol{B}$) to write:

$$\boldsymbol{\nabla}\cdot((\boldsymbol{w}\times\boldsymbol{v})\times\boldsymbol{v}) = \boldsymbol{v}\cdot\boldsymbol{\nabla}\times(\boldsymbol{w}\times\boldsymbol{v}) - \cancel{(\boldsymbol{w}\times\boldsymbol{v})\cdot(\boldsymbol{\nabla}\times\boldsymbol{v})}. \tag{76}$$

Combining these, we obtain the local conservation law (73). Integrating over the flow domain, flow helicity is conserved ($d\mathcal{K}/dt = 0$) provided the flux of the helicity current across the boundary vanishes.

16 Hamiltonian and Poisson brackets for inviscid flow

Having established the conservation of energy in adiabatic, barotropic and incompressible flow in Sect. 15, it is natural to seek a Hamiltonian formulation of fluid mechanics where the energy plays the role of the Hamiltonian. To do this, we need appropriate Poisson brackets that will lead to the equations of motion. It turns out that the relevant PBs among fluid variables $(\rho, \boldsymbol{v}, s, p)$ is noncanonical: there is no separation of physical variables into position-type and momentum-type variables[40]. Such brackets were first proposed by Lev Landau (1941) while attempting a quantum theory of superfluid Helium II [?]. Ironically, Landau's quantum mechanical commutation relations preceded the corresponding classical PBs, which were made precise and generalized to charged fluids (magnetohydrodynamics) by Morrison and Greene in 1980 [?]. Landau arrived at his brackets by treating a fluid as a collection of particles whose positions and momenta satisfied canonical commutation relations. We shall not reproduce Landau's derivation but go straight to the resulting PBs.

Barotropic flow Poisson brackets. For compressible barotropic flow, the nonzero equal-time PBs (up to antisymmetry) among the velocity and density variables are:

$$\{\boldsymbol{v}(\boldsymbol{x}), \rho(\boldsymbol{y})\} = \boldsymbol{\nabla}_{\boldsymbol{y}}\delta(\boldsymbol{x}-\boldsymbol{y}) \quad \text{and} \quad \{v_i(\boldsymbol{x}), v_j(\boldsymbol{y})\} = \frac{\epsilon_{ijk}w_k}{\rho}\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{77}$$

These noncanonical PBs are extended by postulating linearity and the Leibniz rule (see Phys. Plasmas **23**, 022308 (2016) for the Jacobi identity). Being associated with positions and momenta of distinct particles, variables at distinct locations commute. At the same location, the PB can diverge due to the Dirac δ function. While the $\{v_i, v_j\}$ PB being proportional to $\epsilon_{ijk}w_k/\rho$ ensures antisymmetry and the correct dimensions, that between $\boldsymbol{v}$ and ρ is a 'constant' vector, the gradient of $\delta(\boldsymbol{x}-\boldsymbol{y})$ is not dependent on dynamical variables. The appearance of the Dirac δ function may look foreboding, but it is the natural generalization of the Kronecker delta ($\{x_i, p_j\} = \delta_{ij}$) to the fields of continuum mechanics. What is more, in practice, one or other of the spatial variables $\boldsymbol{x}, \boldsymbol{y}$ is integrated after multiplying by some fields, so one may use the defining

[40] Clebsch potentials furnish canonical variables for a fluid, see Sect. 17.

property $\int \delta(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y}) d\boldsymbol{y} = f(\boldsymbol{x})$ to work with them. For instance, let us see how these PBs along with the Hamiltonian for barotropic flow

$$H = \int \left[\frac{1}{2}\rho v^2 + \mathcal{U}(\rho) \right] d\boldsymbol{r} \quad \text{where} \quad \mathcal{U}'(\rho) = h(\rho) \tag{78}$$

lead to the continuity equation (15). Here $h(\rho)$ is the specific enthalpy. Since density commutes with itself, we have

$$\begin{aligned} \partial_t \rho(\boldsymbol{x}) &= \{\rho(\boldsymbol{x}), H\} = \int \frac{\rho(\boldsymbol{y})}{2} \{\rho(\boldsymbol{x}), v(\boldsymbol{y})^2\} d\boldsymbol{y} = \int \rho(\boldsymbol{y}) \boldsymbol{v}_i(\boldsymbol{y}) \{\rho(\boldsymbol{x}), \boldsymbol{v}_i(\boldsymbol{y})\} d\boldsymbol{y} \\ &= \int \rho(\boldsymbol{y}) \boldsymbol{v}_i(\boldsymbol{y}) \partial_{y^i} \delta(\boldsymbol{x}-\boldsymbol{y}) d\boldsymbol{y} = -\int \boldsymbol{\nabla}_{\boldsymbol{y}} \cdot (\rho(\boldsymbol{y}) \boldsymbol{v}(\boldsymbol{y})) \delta(\boldsymbol{x}-\boldsymbol{y}) d\boldsymbol{y} \\ &= -\boldsymbol{\nabla}_{\boldsymbol{x}} \cdot (\rho(\boldsymbol{x}) \boldsymbol{v}(\boldsymbol{x})). \end{aligned} \tag{79}$$

We integrated by parts in the penultimate step to obtain (15). We begin to see how these novel PBs conspire to do their job. A similar calculation leads to the Euler equation (32) in the absence of external body forces.

Recall that the position-momentum PBs $\{x^i, p_j\} = \delta^i_j$ may be expressed as PBs between general observables $\{f(x,p), g(x,p)\} = \sum_i (\frac{\partial f}{\partial x^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p^i} \frac{\partial g}{\partial x_i})$. In a similar vein, (77) may be written as a PB between functionals of ρ and $\boldsymbol{v}$:

$$\{F, G\} = \int \left[\frac{\boldsymbol{w}}{\rho} \cdot \left(\frac{\delta F}{\delta \boldsymbol{v}} \times \frac{\delta G}{\delta \boldsymbol{v}} \right) - \frac{\delta F}{\delta \boldsymbol{v}} \cdot \boldsymbol{\nabla} G_\rho + \frac{\delta G}{\delta \boldsymbol{v}} \cdot \boldsymbol{\nabla} F_\rho \right] d\boldsymbol{r}. \tag{80}$$

Here, subscripts $F_\rho = \frac{\delta F}{\delta \rho}$ denote functional derivatives. In Prob. **??**, we verify that taking $F = \boldsymbol{v}(\boldsymbol{x})$ and $G = \rho(y)$ or $G = \boldsymbol{v}(\boldsymbol{y})$, we recover (77).

17 Clebsch variables and Lagrangian for ideal flow

As we learned in Sect. 16, the Poisson brackets among velocity, density, pressure and specific entropy are not canonical. Thus, we are led to ask whether one can find canonically conjugate fluid variables[41]. It turns out that these are furnished by variables that go back to the work of Alfred Clebsch (1859).

Velocity potential as conjugate to density. To begin with, we notice that the ρ-$\boldsymbol{v}$ equal-time PB $\{\rho(\boldsymbol{x}), \boldsymbol{v}(\boldsymbol{y})\} = \boldsymbol{\nabla}_{\boldsymbol{y}} \delta(\boldsymbol{x}-\boldsymbol{y})$ (77) can be made to look canonical if $\boldsymbol{v}$ happens to be the gradient of a velocity potential ($\boldsymbol{v}(\boldsymbol{y}) = \boldsymbol{\nabla}_{\boldsymbol{y}} \phi(\boldsymbol{y})$) and we postulate that

$$\{\rho(\boldsymbol{x}), \phi(\boldsymbol{y})\} = \delta(\boldsymbol{x}-\boldsymbol{y}), \quad \{\rho(\boldsymbol{x}), \rho(\boldsymbol{y})\} = 0 = \{\phi(\boldsymbol{x}), \phi(\boldsymbol{y})\}. \tag{81}$$

These canonical PBs would then imply that velocity components commute:

$$\{v_i(\boldsymbol{x}), v_j(\boldsymbol{y})\} = \{\partial_{x^i} \phi(\boldsymbol{x}), \partial_{y^j} \phi(\boldsymbol{y})\} = \partial_{x^i} \partial_{y^j} \{\phi(\boldsymbol{x}), \phi(\boldsymbol{y})\} = 0. \tag{82}$$

Pleasantly, this agrees with (77), since vorticity vanishes for potential flow.

[41] By this we mean a collection of fields $q^i(\boldsymbol{r}), p_j(\boldsymbol{r})$, half of which are like position variables and the other half like momentum variables satisfying the PBs $\{q^i(\boldsymbol{r}), p_j(\boldsymbol{r}')\} = \delta^i_j \delta(\boldsymbol{r}-\boldsymbol{r}')$.

Clebsch variables for barotropic flow. Evidently, to deal with flows with vorticity, we need to generalize the formula $\boldsymbol{v} = \boldsymbol{\nabla}\phi$. Such a generalization was found by Clebsch in 1859. Let us first consider the case of homentropic or barotropic flow so that we need not concern ourselves with a dynamical entropy. Clebsch found a way of parametrizing such a velocity field in terms of three 'potentials' ϕ, λ and μ:

$$\boldsymbol{v} = \boldsymbol{\nabla}\phi + (\lambda/\rho)\boldsymbol{\nabla}\mu. \tag{83}$$

Unlike the Helmholtz decomposition of a vector field as the sum of curl-free and divergence-free parts, the Clebsch representation expresses $\boldsymbol{v}$ as a sum of curl-free and helicity-free summands (see Prob. **??**). In a sense, we have traded the three components of $\boldsymbol{v}$ for the three Clebsch potentials[42]. It is also reasonable to have a total of four fields ρ, ϕ, λ and μ, so that they can be split evenly into 'position-type' and 'momentum-type' variables. Later in this section, we will interpret the Clebsch potentials as Lagrange multipliers. The vorticity is given by

$$\boldsymbol{w} = \boldsymbol{\nabla} \times \boldsymbol{v} = \boldsymbol{\nabla}(\lambda/\rho) \times \boldsymbol{\nabla}\mu. \tag{84}$$

Now, we postulate the canonical[43] equal-time PBs among Clebsch variables:

$$\{\rho(\boldsymbol{r}), \phi(\boldsymbol{r}')\} = \{\lambda(\boldsymbol{r}), \mu(\boldsymbol{r}')\} = \delta(\boldsymbol{r} - \boldsymbol{r}'). \tag{85}$$

Up to antisymmetry, the remaining PBs vanish (e.g., $\{\lambda(\boldsymbol{x}), \lambda(\boldsymbol{y})\} = \{\mu(\boldsymbol{x}), \phi(\boldsymbol{y})\} = 0$). Using these, we evaluate

$$\{\rho(\boldsymbol{x}), \boldsymbol{v}(\boldsymbol{y})\} = \{\rho(\boldsymbol{x}), \boldsymbol{\nabla}_{\boldsymbol{y}}\phi(\boldsymbol{y}) + \frac{\lambda}{\rho}\boldsymbol{\nabla}_{\boldsymbol{y}}\mu(\boldsymbol{y})\} = \boldsymbol{\nabla}_{\boldsymbol{y}}\{\rho(\boldsymbol{x}), \phi(\boldsymbol{y})\} = \boldsymbol{\nabla}_{\boldsymbol{y}}\delta(\boldsymbol{x} - \boldsymbol{y}), \tag{86}$$

which is as desired. The velocity-velocity PB also agrees with (77):

$$\{v_i(x), v_j(y)\} = \frac{1}{\rho}\left(\partial_i\left(\frac{\lambda}{\rho}\right)\partial_j\mu - \partial_j\left(\frac{\lambda}{\rho}\right)\partial_i\mu\right)\delta(x-y) = \frac{1}{\rho}\epsilon_{ijk}w_k\delta(x-y). \tag{87}$$

The arguments of the functions multiplying $\delta(x-y)$ can be taken as x or y. Thus, the Clebsch variables furnish canonical (or Darboux) coordinates for barotropic flow.

Hamiltonian. The Hamiltonian in terms of Clebsch variables is

$$H = \int \mathcal{H}\, d\boldsymbol{r} = \int \left[\frac{\rho}{2}\left(\boldsymbol{\nabla}\phi + \frac{(\lambda\boldsymbol{\nabla}\mu)}{\rho}\right)^2 + \rho\,\varepsilon(\rho)\right] d\boldsymbol{r}. \tag{88}$$

Here, $\mathcal{U} = \rho\varepsilon$ is the potential energy density and $\varepsilon(\rho)$ the specific internal energy. Hamilton's equations $\partial_t f = \{f, H\}$ for $f = \rho, \boldsymbol{v}$ then reproduce the continuity and Euler equations as in Sect. 16.

[42] However, the Clebsch potentials are not uniquely determined by $\boldsymbol{v}$ and ρ. To begin with, we may add constants to ϕ and μ without altering $\boldsymbol{v}$. In fact, there are more 'gauge transformations' one can make without affecting $\boldsymbol{v}$.

[43] The division of λ by ρ in (83) is to ensure that λ and μ may be taken to be canonically conjugate.

Lagrangian and equations of motion. The advantage of having Clebsch variables is that they can be used to give a Lagrangian formulation for the equations of inviscid flow. Since the equations of motion are first order in time derivatives, the relevant Bateman-Thellung Lagrangian density $\mathcal{L}_1 = \rho_t\phi + \lambda_t\mu + \alpha_t s - \mathcal{H}$ is linear (rather than quadratic) in generalized velocities

$$\mathcal{L}_1 \quad = \quad \rho_t\phi + \lambda_t\mu + \alpha_t s - \frac{\rho}{2}\left(\boldsymbol{\nabla}\phi + \frac{(\lambda\boldsymbol{\nabla}\mu + \alpha\boldsymbol{\nabla}s)}{\rho}\right)^2 - \rho\,\varepsilon(\rho). \tag{89}$$

$\mathcal{L}_1$ depends on the 4 fields $f = \rho, \phi, \lambda, \mu$ and their space or time derivatives. The Euler-Lagrange (EL) equation for a field f is given by

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}_1}{\partial f_t} = \frac{\partial \mathcal{L}_1}{\partial f} - \boldsymbol{\nabla}\cdot\frac{\partial \mathcal{L}_1}{\partial \boldsymbol{\nabla} f}. \tag{90}$$

For example, the EL equation for λ implies that μ is advected[44]: $\frac{D\mu}{Dt} = 0$. On the other hand, $\mathcal{L}_1$ is independent of ϕ_t while $\partial\mathcal{L}_1/\partial\phi = \rho_t$ and $\partial\mathcal{L}_1/\partial\boldsymbol{\nabla}\phi = -\rho\boldsymbol{v}$. Thus, the EL equation for ϕ is the continuity equation $\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0$. Proceeding as above, the EL equation for μ says that λ is locally conserved: $\lambda_t + \boldsymbol{\nabla}\cdot(\lambda\boldsymbol{v}) = 0$. The EL equation for ρ is quite interesting. We have $\partial\mathcal{L}_1/\partial\rho_t = \phi$ and

$$\begin{aligned}\frac{\partial \mathcal{L}_1}{\partial \rho} \quad &= \quad -\frac{1}{2}v^2 - \frac{\rho}{2}2\boldsymbol{v}\cdot\left(-\frac{1}{\rho^2}(\lambda\boldsymbol{\nabla}\mu)\right) - \varepsilon - \rho\frac{\partial\varepsilon}{\partial\rho} \\ &= \quad -\frac{1}{2}v^2 + \boldsymbol{v}\cdot(\boldsymbol{v} - \boldsymbol{\nabla}\phi) - \varepsilon - \frac{p}{\rho},\end{aligned} \tag{91}$$

upon using $p = \rho^2\partial\varepsilon/\partial\rho$. Thus, we get a sort of time-dependent Bernoulli equation:

$$\phi_t - \frac{\boldsymbol{v}^2}{2} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\phi + \varepsilon(\rho) + \frac{p}{\rho} = 0. \tag{92}$$

Taking its gradient and combining with the other equations, one obtains the Euler equation for $\boldsymbol{v}$ (22), see Prob. **??**.

18 Obtaining the heat equation from Fourier's law

Suppose we have a body with some initial (absolute) temperature[45] distribution $T(\boldsymbol{r}, t = 0)$. We wish to know how this temperature distribution evolves with time. Empirically, it is found that the heat flux between bodies or parts of a body grows with the temperature difference. Fourier's law of heat diffusion states that the heat flux density vector (with units of energy per unit time crossing unit area normal to the heat flux vector) is proportional to the negative gradient in temperature

$$\boldsymbol{q} = -k\boldsymbol{\nabla}T \quad \text{where } k \text{ is the thermal conductivity.} \tag{93}$$

[44] This means that μ of a fluid element is the same as at $t = 0$

[45] Absolute temperature is defined through the 2^{nd} law of thermodynamics. The Kelvin scale is a scale of absolute temperature.

For a perfect thermal insulator, k would be zero. If $d\boldsymbol{S} = \hat{\boldsymbol{n}} dS$ is a small area (vector), then the heat flux across it (energy crossing it per unit time) is given by $\boldsymbol{q} \cdot d\boldsymbol{S}$.

Consider gas in a fixed volume V. The increase in internal energy

$$U = \int_V \rho c_v T(\boldsymbol{r}, t)\, d\boldsymbol{r}, \qquad (94)$$

where T is the absolute temperature, must be due to the influx of heat across its surface ∂V. This is a consequence of the first law of thermodynamics, if we assume no work is done on the gas[46] and that there are no sources/sinks of energy inside the gas. Thus

$$\int_V \partial_t(\rho c_v T) d\boldsymbol{r} = -\int_{\partial V} \boldsymbol{q} \cdot \hat{\boldsymbol{n}}\, dS = \int_{\partial V} k \boldsymbol{\nabla} T \cdot \hat{\boldsymbol{n}}\, dS = k \int_V \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} T\, d\boldsymbol{r}. \qquad (95)$$

We used Gauss' divergence theorem to convert the surface integral to a volume integral taking $\hat{\boldsymbol{n}}$ to be the outward pointing normal to the surface. Here, c_v is the specific heat per unit mass at constant volume (since no work is done) and ρ is the density of the gas. Since the volume V is arbitrary, the integrands must be equal and Fourier's heat diffusion equation follows:

$$\frac{\partial T}{\partial t} = \alpha \boldsymbol{\nabla}^2 T \quad \text{where} \quad \alpha = \frac{k}{\rho c_v} \quad \text{is called the thermal diffusivity.} \qquad (96)$$

The diffusivity $\alpha \geq 0$ has dimensions of area per unit time[47]. Since the heat equation is linear and only involves derivatives of temperature, we are free to rescale the absolute temperature or add a constant to it. Consequently, the heat equation applies even if we use the Centigrade or Fahrenheit scales in place of an absolute temperature scale such as the Kelvin scale.

19 Navier-Stokes equation for incompressible viscous flow

If viscous dissipative effects are included, the Euler equation (22) for inviscid flow is modified. The simplest case is that of incompressible flow, where (22) is augmented by a viscous term $\propto \boldsymbol{\nabla}^2 \boldsymbol{v}$ to obtain the Navier-Stokes (NS) equation[48]:

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = -\boldsymbol{\nabla} p + \boldsymbol{f} + \mu \boldsymbol{\nabla}^2 \boldsymbol{v} \quad \text{with} \quad \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0. \qquad (98)$$

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[46] The first law of thermodynamics for a gas states that $\Delta U = \Delta Q + \Delta W$. Here, ΔU is the increase in internal energy of the gas, and it is a sum of the heat added ΔQ and the work ΔW done on the gas. If no work is done and there are no sources of heat in the interior, then ΔU must be due to the heat transferred across the boundary.

[47] The diffusion equation can be used to describe the diffusion of heat, material/molecules, momentum, velocity, vorticity, etc. In each case, where an analog of Fourier's law holds, there is an analog of the thermal conductivity k, but it has a dimension depending on context. By contrast, diffusivities such as α always have dimensions of an areal speed and can be compared to know the relative rates of diffusion.

[48] Using a vector identity, the advection term in NS may be written in terms of the Lamb vector:

$$\partial_t \boldsymbol{v} + \boldsymbol{w} \times \boldsymbol{v} = -\frac{1}{\rho} \boldsymbol{\nabla} p - \frac{1}{2} \boldsymbol{\nabla} \boldsymbol{v}^2 + \frac{\boldsymbol{f}}{\rho} + \nu \boldsymbol{\nabla}^2 \boldsymbol{v}. \qquad (97)$$

Here, $\boldsymbol{f}$ is the body force per unit volume and μ is called the dynamic viscosity, which has been assumed constant[49]. One also introduces the kinematic viscosity $\nu = \mu/\rho$ with dimensions of a diffusivity [areal speed (L^2/T)]. If density variations are insignificant, then ν may also be taken to be a constant, as we shall do wherever convenient. The word shear[50] viscosity is often used for ν since, as we shall soon see, it is the coefficient of friction between layers of fluid that slide over each other.

The viscous term can be motivated through an analogy with the heat equation $\partial_t T = \alpha \boldsymbol{\nabla}^2 T$ (96) which describes diffusion of heat from hot to cold regions. Similarly, shear viscosity causes diffusion of velocity from a fast layer to a neighboring slow layer of fluid. By analogy with heat diffusion, velocity diffusion is described by $\nu \boldsymbol{\nabla}^2 \boldsymbol{v}$, with shear viscosity ν playing a role analogous to thermal diffusivity α. If water in a cup is stirred and left to itself, viscosity brings it to rest, just as heat diffusion uniformizes the temperature distribution to zero in a rod with ends held at $T = 0$.

The NS equation has not been derived from molecular dynamics except for dilute gases. It is the simplest equation consistent with physical requirements and symmetries, that can be used to describe macroscopic viscous fluid motion. As with other physical models, its validity is to be checked by comparing its predictions with experimental measurements, which have largely confirmed its reliability.

No-slip boundary condition. The Navier-Stokes equation (98) is $2^{\rm nd}$ order in space derivatives unlike the inviscid Euler equation (22), which is $1^{\rm st}$ order. The viscous term is called a singular perturbation, it increases the spatial order by one and necessitates an additional boundary condition (BC). In addition to the impenetrable BC on solid boundaries[51], one typically imposes the 'no-slip' BC which requires that the tangential component of $\boldsymbol{v}$ on solid boundaries must vanish. There is empirical support for the no-slip boundary condition: (a) running a fan does not remove the dust accumulated on its blades and (b) material accumulates on the sides of drain pipes even if the flow is quite fast.

Stress and rate of strain tensors. As we did with the Euler equation in (21), we may write the incompressible NS equation (97) in the absence of external body forces in terms of a stress tensor:

$$\partial_t v_i + v_j \partial_j v_i = -\frac{1}{\rho} \partial_j T_{ij} \quad \text{where} \quad T_{ij} = p\delta_{ij} - \mu(\partial_i v_j + \partial_j v_i). \tag{99}$$

The term $\partial_i v_j$ in T_{ij} does not contribute to the EOM through the divergence of the stress tensor $\partial_j T_{ij}$ because the flow is incompressible ($\partial_j v_j = 0$). However, we

[49]The dynamic viscosity μ can vary with location, especially if there are significant inhomogeneities in temperature. If μ is nonuniform, then the $i^{\rm th}$ component of the viscous term in (98) becomes $\partial_j(\mu \partial_j v_i)$, see (99).

[50]The other type of viscosity is bulk or volume viscosity, which has to do with friction from compression of the fluid. It is absent in incompressible flows.

[51]The impenetrable BC is $\boldsymbol{v} \cdot \hat{\boldsymbol{n}} = 0$, where $\hat{\boldsymbol{n}}$ is the outward normal on fixed boundaries.

include it to make T_{ij} a symmetric tensor[52]. Evidently, pressure[53] enters through the isotropic part[54] (part that is a multiple of the identity) while the viscous forces enter through its nonisotropic part (traceless part: $-2\mu\partial_i v_i = 0$)[55]. This reinforces our introduction of pressure as a normal surface force, while shear viscosity is a tangential surface force. The (negative of the) nonisotropic part of T_{ij} is denoted d_{ij} and is called the deviatoric stress tensor. It is the part of the stress that can cause a fluid element to change shape but not volume. It must be symmetric and traceless: $d_{ij} = d_{ji}$ and $d_{ii} = 0$. For an incompressible viscous fluid, we have postulated that $d_{ij} = \mu(\partial_i v_j + \partial_j v_i)$.

We notice that the deviatoric stress can be nonzero only if the fluid is in motion ($\boldsymbol{v} \neq 0$). In fact, one needs more than just a flow, one needs appropriate velocity gradients to be nonvanishing. It is convenient at this stage to introduce the rate of strain tensor[56] $e_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$, in terms of which, our postulate becomes $d_{ij} = 2\mu e_{ij}$. This is an instance of Newton's law for a fluid, stating that the deviatoric stress is linearly related to (here proportional to) the rate of strain[57]. In fact, one could start more generally by postulating that the deviatoric stress is linear[58] in velocity gradients $d_{ij} = A_{ijkl}\partial_k v_l$ for some constant 4^{th} rank tensor A_{ijkl}, and use the assumption of isotropy of the fluid[59] and the symmetry of d_{ij} to show that one must have $d_{ij} = 2\mu e_{ij}$ for an incompressible flow. The tensor A_{ijkl}, being an intrinsic property of the fluid must be an isotropic tensor. What this means is that its components must be the same irrespective of which Cartesian frame is used. Rotating the frame must not change the

[52] Taking T_{ij} symmetric could be regarded as an attempt at being elegant without affecting the equation of motion. However, it is helpful elsewhere. E.g., Batchelor (p.11 of An Introduction to Fluid Dynamics (2000)) argues that the stress tensor must be symmetric for the rate of change of angular momentum of a fluid element to equal the torque on the element due to surface forces. It also needs to be symmetric if the fluid (say, in a gaseous star) is to be consistently coupled to (i.e., act as a source for) the gravitational field in Einstein's general theory of relativity.

[53] As in the incompressible Euler equation, the pressure may be eliminated via the constraint equation obtained by taking the divergence of the NS equation and using $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ (see (24)).

[54] An isotropic second rank tensor t_{ij} is one that does not define any preferred direction. Eigenvectors of the matrix t_{ij} define preferred directions. So for a matrix to be isotropic, every vector must be an eigenvector. This is the case if it is a multiple of the identity. Since multiples of the identity are invariant under changes of basis, this concept is basis-independent.

[55] Sometimes, it is imprecisely said that pressure is the diagonal part and the viscous forces are the off-diagonal part of the stress tensor: this is a basis-dependent statement.

[56] The velocity gradient $\partial_i v_j$ can be written as a sum of its symmetric and antisymmetric parts $\partial_i v_j = e_{ij} + w_{ij}$ where $e_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$ is the rate of strain tensor and $w_{ij} = \frac{1}{2}(\partial_i v_j - \partial_j v_i)$ is the vorticity tensor: $w_i = \epsilon_{ijk} w_{jk}$. The name *rate of strain* is because $e_{ij} \sim \frac{\partial}{\partial x_i}\frac{Dx_j}{Dt}$ involves a time derivative (rate) as well as a fractional displacement (strain).

[57] This is the Newtonian fluid analog of Hooke's law for an elastic solid: stress being proportional to strain or more precisely, the stress tensor being linearly related to the strain tensor.

[58] A linear relation between two vectors (e.g., $L_i = \mathcal{I}_{ij}\Omega_j$) involves a 2^{nd} rank tensor. Similarly, a linear relation between two second rank tensors involves a fourth rank tensor.

[59] A fluid is isotropic if its intrinsic properties are independent of the direction in which they are measured. A liquid crystal with elongated molecules that are aligned using electromagnetic fields is an anisotropic fluid, as are some states of polymers. By intrinsic properties, we mean things like the refractive index or shear viscosity and not the shape of the container or features of a particular flow of the fluid, which may have a specific direction. Isotropy of a fluid implies that if it admits a flow in one direction, it would admit the same type of flow in any other direction (with suitably oriented boundaries).

components. We are familiar with this concept for 2nd rank tensors: A_{ij} is isotropic iff it is a multiple of the identity $A_{ij} = \lambda \delta_{ij}$. The eigenvectors of a matrix define special directions. For isotropy, all directions must be eigendirections. This is possible only for multiples of the identity. The only isotropic tensor of rank one (vector) is the zero vector, which points in all directions. The only isotropic tensors of rank 3 are multiples of the Levi-Civita symbol ϵ_{ijk}.

More general isotropic tensors may be obtained via linear combinations of products of the Kronecker delta and Levi-Civita symbols. In particular, we may express the isotropic 4th rank tensor A_{ijkl} as

$$A_{ijkl} = a\delta^i_j \delta^k_l + b\delta^i_k \delta^j_l + c\delta^i_l \delta^j_k. \tag{100}$$

Newton's law then becomes

$$d_{ij} = (a\delta^i_j \delta^k_l + b\delta^i_k \delta^j_l + c\delta^i_l \delta^j_k)\partial_k v_l = b\partial_i v_j + c\partial_j v_i. \tag{101}$$

Since $d_{ij} = d_{ji}$ should be symmetric, $b = c$. Thus $d_{ij} = 2be_{ij}$. Taking $b = \mu$ we recover the proportionality relation between deviatoric stress and rate of strain $d_{ij} = 2\mu e_{ij}$.

This completes our analogy between heat and velocity diffusion. Just as the heat flux vector (93) is proportional to the temperature gradient, the deviatoric viscous stress is proportional to the velocity gradient.

Deviatoric stress for shear flow. To get a feeling for the deviatoric stress, let us consider a horizontal shear flow with $\boldsymbol{v} = (u(y), 0, 0)$. In this case, the only independent deviatoric stress tensor component is

$$T_{12} = T_{21} = -d_{12} = -2\mu e_{12} = -\mu(\partial_1 v_2 + \partial_2 v_1) = -\mu u'(y). \tag{102}$$

Evidently, this viscous stress arises from the relative motion of horizontal layers.

Poiseuille flow in a pipe. This is steady laminar flow of a viscous fluid with constant density ρ through a long horizontal cylindrical pipe of length ℓ and uniform circular cross section of radius $a \ll \ell$, with axis along the z-axis. It is induced by a pressure drop Δp between the inlet and outlet of the pipe, with effects of gravity being insignificant. For sufficiently small[60] pressure gradients and away from the inlet and outlet, the flow is found to settle into a steady axisymmetric pattern. It is an example of a shear flow: faster at the center of the pipe and tapering off to zero on the walls, where it satisfies the no-slip BC. We will work in cylindrical coordinates and suppose that the flow velocity is independent of z and is purely axial $\boldsymbol{v} = u(r)\hat{z}$, while the pressure could depend on r and z. The time derivative and advection terms vanish in the incompressible NS equation $\rho(\boldsymbol{v}_t + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}) = -\boldsymbol{\nabla} p + \mu \boldsymbol{\nabla}^2 \boldsymbol{v}$ as the flow is steady and lacks a radial velocity component. Moreover, the radial component of NS implies $\partial p/\partial r = 0$, so $p = p(z)$. Separating variables in the z-component of NS,

$$\mu \boldsymbol{\nabla}^2 \boldsymbol{v} = \frac{\mu}{r}\partial_r(r\partial_r u(r))\hat{z} = \frac{\partial p}{\partial z}\hat{z}, \tag{103}$$

[60] As Δp is increased, the flow becomes unsteady and makes a transition to turbulence.

we infer that the pressure gradient must be a constant, $dp/dz = -\Delta p/\ell < 0$. Upon imposing the BC $u(a) = 0$, we find a parabolic radial profile for the longitudinal velocity and an azimuthal vorticity that is counterclockwise viewed from the outlet:

$$u(r) = \frac{\Delta p}{4\mu\ell}(a^2 - r^2) \quad \text{and} \quad \boldsymbol{w} = -u'(r)\hat{\phi} = \frac{r\Delta p}{2\mu\ell}\hat{\phi}. \tag{104}$$

Navier-Stokes for compressible flow. For compressible flow, there are two types of viscous terms, essentially because there are two vectors that can be constructed from second derivatives of velocity: the Laplacian $\boldsymbol{\nabla}^2 \boldsymbol{v}$ and the gradient of the divergence $\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{v})$. For uniform dynamic shear viscosity μ and bulk viscosity ζ, the compressible Navier-Stokes equation takes the form [**?**]

$$\rho \frac{D\boldsymbol{v}}{Dt} = \boldsymbol{f} - \boldsymbol{\nabla} p + \mu \left(\boldsymbol{\nabla}^2 \boldsymbol{v} + \frac{1}{3} \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{v}) \right) + \zeta \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{v}). \tag{105}$$

Here p is the mechanical pressure, the isotropic part of the stress tensor. In a flowing viscous fluid, the mechanical pressure can differ from the thermodynamic pressure that appears in the thermodynamic equation state. □

Vorticity evolution and diffusion. The equation for evolution of vorticity may be obtained by taking the curl of (97). For a *conservative* body force and *constant density*, it takes a particularly simple form:

$$\partial_t \boldsymbol{w} + \boldsymbol{\nabla} \times (\boldsymbol{w} \times \boldsymbol{v}) = \nu \boldsymbol{\nabla}^2 \boldsymbol{w} \quad \text{where} \quad \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0. \tag{106}$$

We see that each component of vorticity must satisfy the linear diffusion equation ($\partial_t \boldsymbol{w} = \nu \boldsymbol{\nabla}^2 \boldsymbol{w}$) modified by the curl of the nonlinear vortex force. Thus, unlike in inviscid flow studied in Sect. 13 and Sect. 14, vorticity can diffuse between regions and is not simply frozen into the velocity field. In particular, vortex tubes can lose their strength with time and dissipate. In viscous flows, vorticity is often generated near boundaries (even if not initially present) and then diffuses to the bulk of the fluid.

Reynolds number and similarity principle. Suppose we consider water with uniform velocity $U\hat{x}$ flowing along a broad and deep channel. It meets a cylindrical obstacle of diameter L and flows round it creating a pattern as in Fig. 8a. It turns out that if we double the speed U and halve the diameter L, then the same flow pattern results. This is the 'similarity' principle named after Osborne Reynolds, who did careful experiments with fluids flowing through a pipe in the late 1800s. More precisely, incompressible flows[61] with the same Reynolds number $\mathcal{R}$ are identical when compared at a reference scale. Reynolds' similarity principle is quite useful in practice: it is exploited to study the flow around an aircraft by using a flow with the same $\mathcal{R}$ around a scaled-down version of the aircraft placed in a wind tunnel, leading to significant cost savings.

[61] The flows could be steady or unsteady, laminar or turbulent.

To see how this arises, let us use the scales L and U (which are an appropriate geometric length and typical flow speed) to define dimensionless (primed) variables

$$\boldsymbol{r}' = \boldsymbol{r}/L, \quad \boldsymbol{v}' = \boldsymbol{v}/U \quad \text{and} \quad t' = Ut/L. \tag{107}$$

We denote by $\boldsymbol{\nabla}'$, the gradient with respect to $\boldsymbol{r}'$. Then, the Navier-Stokes equation (106) for incompressible constant density flow (in vorticity form) becomes

$$\frac{\partial \boldsymbol{w}'}{\partial t'} + \boldsymbol{\nabla}' \times (\boldsymbol{w}' \times \boldsymbol{v}') = \left(\frac{\nu}{LU}\right) \boldsymbol{\nabla}'^2 \boldsymbol{w}' \quad \text{with} \quad \boldsymbol{\nabla}' \cdot \boldsymbol{v}' = 0, \tag{108}$$

where $\boldsymbol{w}' = \boldsymbol{\nabla}' \times \boldsymbol{v}'$. Thus, two flows with the same Reynolds number $\mathcal{R} = LU/\nu$ lead to the same equation, and are therefore simply rescaled versions of each other.

The Reynolds number $\mathcal{R}$ is a dimensionless parameter that may be interpreted as a measure of the ratio of inertial to viscous terms in the NS equation (98):

$$\frac{\text{inertial force}}{\text{viscous force}} = \frac{|\boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}|}{|\nu \boldsymbol{\nabla}^2 \boldsymbol{v}|} \approx \frac{U^2/L}{\nu U/L^2} = \frac{LU}{\nu} = \mathcal{R}. \tag{109}$$

When $\mathcal{R}$ is small (e.g., in slow creeping flow relevant to swimming microbes (E M Purcell, Life at low Reynolds number)), viscous forces dominate over the nonlinear inertial forces and the flow is regular or laminar, or even steady [$\boldsymbol{v}(\boldsymbol{r}, \not{t})$ and $p(\boldsymbol{r}, \not{t})$ independent of time when there is a suitable driving force to balance dissipation]. At low $\mathcal{R}$, one could ignore the nonlinear inertial advection term $\boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}$ in NS, resulting in the Stokes flow approximation. On the other hand, when $\mathcal{R}$ increases (say, as the flow speeds up), the streamlines become convoluted, the flow becomes increasingly irregular, seemingly unpredictable and is called turbulent.

Stokes flow past a sphere. Stokes studied steady creeping viscous constant ρ flow with asymptotic[62] velocity $\boldsymbol{U}$ around a sphere of radius a. For steady ($\partial_t \boldsymbol{v} = 0$) creeping flow ($\mathcal{R} = aU/\nu \ll 1$), (98) reduces to the linear equation

$$0 = -\boldsymbol{\nabla} p + \mu \boldsymbol{\nabla}^2 \boldsymbol{v} \quad \text{with} \quad \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0. \tag{110}$$

The dimensional parameters in the problem are U, a and $\mu = \rho\nu$. Thus, on dimensional grounds, the magnitude of the force that the fluid exerts on the sphere must be $F \propto \mu a U$. A detailed calculation (see Sect. 20 of Landau & Lifshitz, Fluid mechanics) shows that the proportionality constant is 6π. In other words, the drag force on a sphere moving at velocity $\boldsymbol{U}$ through a fluid asymptotically at rest is $-6\pi a \mu \boldsymbol{U}$ in the Stokes flow approximation. More generally, even if we drop the assumptions of steady creeping flow, dimensional analysis implies that the magnitude of the drag force may be written as $F = \frac{1}{2} C_D(\mathcal{R}) \pi a^2 \rho U^2$ for a dimensionless function $C_D(\mathcal{R})$, called the drag coefficient. What the Stokes approximation shows is that $C_D \to 12/\mathcal{R}$ as $\mathcal{R} \to 0$. This has been experimentally validated. In other words, the drag force is proportional to speed at low speeds. As $\mathcal{R}$ is increased, there are deviations from

[62]By going to a frame that moves with the asymptotic velocity of the fluid, the problem may be mapped to that of a sphere moving at $-\boldsymbol{U}$ through a fluid asymptotically at rest.

Stokes' formula as the flow ceases to be steady. It becomes roughly periodic in time (but still laminar) and then becomes increasingly turbulent: vortices are generated in a boundary layer around the sphere and are carried downstream in a turbulent wake. This change in flow pattern with increasing $\mathcal{R}$ is described below in the context of flow past a cylinder.

Flow past a cylinder: transition to turbulence[63]**.** Let us consider flow of water, entering uniformly from the far left and flowing to the right with asymptotic velocity $U\hat{x}$ as in Fig. 8a. The water meets a fixed vertical right circular cylinder of diameter L with axis along $\hat{z}$. We shall denote the Cartesian components of the flow velocity by $\boldsymbol{v} = (u, v, w)$. At very low $\mathcal{R} = LU/\nu \approx .16$, the flow around the cylinder is laminar (steady, i.e., time-translation invariant) and displays several symmetries, as shown in Fig. 8a: (a) $y \to -y$ (reflections in $z - x$ plane), (b) z translation-invariance and (c) left-right symmetry with respect to the center of the cylinder ($x \to -x$ and $(u, v, w) \to (u, -v, -w)$). All these are symmetries of Stokes flow, which results from ignoring the body force term and nonlinear advection term in NS (98). At $\mathcal{R} \approx 1.5$, a marked left-right asymmetry develops between the upstream and downstream regions. At $\mathcal{R} \approx 5$, there is a change in the topology of the flow: the flow downstream of the obstacle no longer hugs the cylindrical surface, it detaches from the surface. This is called flow separation and is associated with the formation of recirculating standing eddies downstream of the cylinder (see Fig. 8b). These eddies were not present at smaller $\mathcal{R}$ or far upstream of the cylinder; they are generated and diffuse from a *boundary layer* around the obstacle, where the effect of viscosity is very significant. Far from the obstacle, the flow is nearly irrotational.

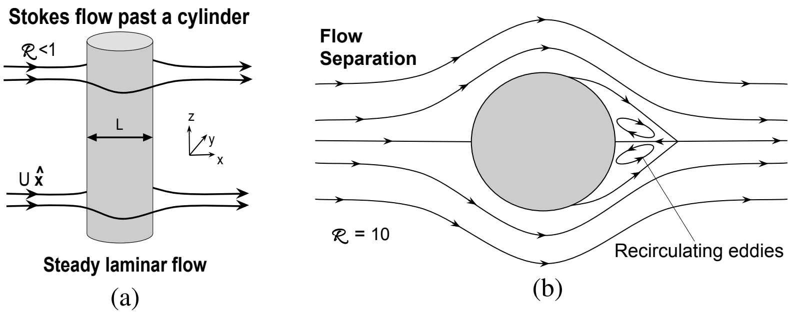

Figure 8: Flow past a cylinder. (a) Side view of Stokes flow at $\mathcal{R} \lesssim 1$. (b) Top view at $\mathcal{R} \approx 10$ showing detachment of the flow and recirculating eddies downstream of the cylinder.

At $\mathcal{R} \approx 40$, the flow ceases to be approximately steady, but is periodic in time at each point. At $\mathcal{R} \gtrsim 40$, recirculating eddies are periodically shed (alternatively from either side of the cylinder) to form the celebrated von Kármán vortex street[64]

[63] Excellent photographs illustrating the transition to turbulence in flow past a cylinder may be found in the book by van Dyke.

[64] Vortex streets also appear behind an obstacle in a river, in clouds passing around a high pressure region, past the wings of insects and birds, etc.

sketched in Fig. 9a. The z-translation invariance along the axis of the cylinder is spontaneously broken when $\mathcal{R} \sim 40 - 75$. At higher $\mathcal{R} \sim 200$, the flow becomes chaotic with a turbulent 'boundary layer' around the cylinder. At $\mathcal{R} \sim 1800$, only about two vortices in the von Kármán vortex street are distinct, before merging into a quasiuniform turbulent wake (see Fig. 9b). At much higher $\mathcal{R}$, many of the symmetries of NS are restored in a statistical sense and turbulence is called fully-developed. This spontaneous breaking and statistical restoration of symmetries of the equations and boundary conditions is typical of the transition from laminar flow to fully developed turbulence.

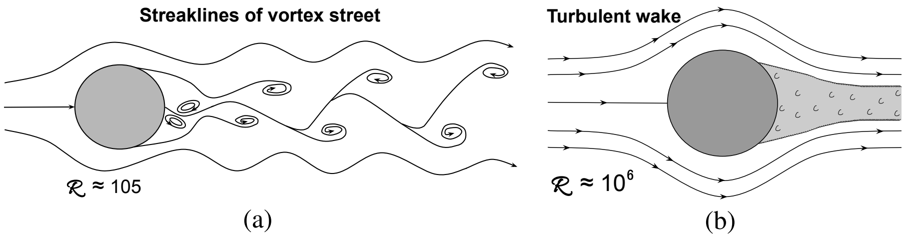

Figure 9: Flow past a cylinder. (a) Top view of streaklines at $\mathcal{R} \approx 105$ showing alternate shedding of eddies and development of vortex street consisting of two parallel rows of staggered vortices. (b) Sketch of turbulent wake downstream of cylinder at $\mathcal{R} \approx 10^6$.

By turbulence, we usually mean irregular[65] behavior (chaos) in a driven dissipative system with a large number of degrees of freedom. Without a driving force (say stirring or pumping in the water), the turbulence decays. In the absence of dissipation, we would have Hamiltonian chaos modeled, for instance, by the Euler equation. In a turbulent flow, the velocity field $\boldsymbol{v}(\boldsymbol{r}, t)$ appears random in time and highly disordered in space. Turbulent flows exhibit a wide range of length scales[66]: from the system size, size of obstacles, through large vortices down to the smallest ones at the so-called Taylor microscale where dissipation occurs. What is more, even at a fixed location $\boldsymbol{r}_0$, the time series of velocity $\boldsymbol{v}(\boldsymbol{r}_0, t)$ tends to be very different in distinct experiments with approximately the same initial and boundary conditions. Nevertheless, the time average $\bar{\boldsymbol{v}}(\boldsymbol{r}_0)$ tends to be the same in all realizations. Unlike individual flow realizations, statistical properties of turbulent flow are empirically reproducible. They typically depend on $\mathcal{R}$ and BCs.

The study of fluid flow including the Navier-Stokes and Euler equations is a major branch of engineering, physics and mathematics. Though simple to write down, these equations are notoriously hard to solve in most physically interesting situations, primarily due to the nonlinearities arising from the advection term and the resulting coupling of a large number of degrees of freedom and length scales. For instance, we do not yet have an effective analytical method of predicting the features of the flow

[65]This includes intermittency, one of whose manifestations is irregular alteration between chaos and apparently periodic behavior.

[66]The range of length scales where inertial forces ($\propto \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}$) dominate over viscous forces ($\propto \nu \boldsymbol{\nabla}^2 \boldsymbol{v}$) is called the inertial range. Dissipation typically occurs at smaller length scales.

past a cylinder as $\mathcal{R}$ is varied or the values of $\mathcal{R}$ at which qualitative changes[67] in the flow pattern occur. However, some things have been done: von Kármán showed that the vortex street is stable for certain ranges of parameters. Numerical simulations can reproduce some of the observed features. As happened near the cylinder, the conditions at boundaries and interfaces encode important physical effects, but can add to the complications. Some of these effects can be understood using boundary layer theory, an asymptotic approximation method pioneered by Ludwig Prandtl. In fact, there is a million dollar Clay Millennium Prize attached to understanding whether the solution to NS for smooth initial data exists and is smooth or whether the solution could hit a singularly in finite time. The wider challenge lies in deducing the observed, often complex patterns of flow[68] from the known laws governing fluid motion. This often requires a mix of physical insight, experimental data, mathematical techniques and computational methods.

[67] Qualitative changes typically occur when the existing flow is unstable to perturbations which grow, leading to a new flow pattern. For instance, steady shear flow can be susceptible to the Kelvin-Helmholtz instability: the interface between two layers of fluid moving at different velocities is unstable to short wavelength perturbations that grow, leading, for instance, to the 'roll-up' of a vortex sheet. Another common instability is the Rayleigh-Taylor instability of the interface between a dense fluid and a rare fluid that lies beneath it.

[68] In particular, one would like to develop a statistical understanding of turbulent flows.