Fluid Dynamics, Autumn 2024, CMI
Assignment 11
Due by 5pm on Friday, Nov 29, 2024
Helmholtz decomposition, Hamiltonian formulation, Navier-Stokes, isotropic tensors

1. $\langle \mathbf{5+5} \rangle$ Consider the Helmholtz decomposition of a given steady velocity vector field $\boldsymbol{v} = \nabla\phi + \nabla \times \boldsymbol{A}$ in $\mathbb{R}^3$. (a) Use the freedom to add to $\boldsymbol{A}$ the gradient of a scalar function χ to show that we may take $\nabla \cdot \boldsymbol{A} = 0$. This is called the Coulomb gauge condition. Find a differential equation for the scalar function and an integral representation of its solution. (b) Find a differential equation and integral formula for $\boldsymbol{A}$ in the Coulomb gauge. You may assume that $\boldsymbol{v}$ vanishes sufficiently fast at infinity.

2. $\langle \mathbf{2+4+5} \rangle$ (a) Write down the Poisson brackets among Clebsch variables ρ, ϕ, λ, μ for barotropic flow governed by the Hamiltonian
$$H = \int \mathcal{H}\, d\boldsymbol{r} = \int \left[\frac{\rho}{2}\left(\nabla\phi + (\lambda/\rho)\nabla\mu\right)^2 + \mathcal{U}(\rho)\right] d\boldsymbol{r}. \quad (1)$$
(b) Find Hamilton's equation for ρ and show that it reduces to the continuity equation. (c) Show that Hamilton's equation for ϕ is a generalization of the time-dependent Bernoulli equation:
$$\phi_t = \frac{1}{2}\boldsymbol{v}^2 - \mathcal{U}'(\rho) - \boldsymbol{v}\cdot\nabla\phi. \quad (2)$$
You may assume decaying boundary conditions and ignore surface terms.

3. $\langle \mathbf{5} \rangle$ In a nuclear explosion, a spherically symmetric expanding blast wave is produced. It may be argued that to first approximation, the radius of this blast wave $R(t)$ can only depend on the total energy E of the explosion, the ambient mass density ρ of the gas into which the blast wave propagates and on time. Find the form of the radius $R(t)$.

4. $\langle \mathbf{3+2+2+3+1} \rangle$ Consider steady slow Poiseuille flow of a viscous fluid with constant density ρ and dynamic viscosity μ through a long horizontal cylindrical pipe of length ℓ and uniform circular cross section of radius $a \ll \ell$, with axis along the z-axis. It is induced by a pressure drop Δp between the inlet and outlet of the pipe, with effects of gravity being insignificant. (a) What is the mass flow rate $\dot{M}$ (mass of fluid passing through a cross section of the pipe per unit time)? (b) What is the corresponding volume flow rate $\dot{V}$? (c) What is the vorticity in Poiseuille flow? (d) Show that the vorticity satisfies the vector Laplace equation $\nabla^2 \boldsymbol{w} = 0$. Why is this to be expected? (e) What is the helicity density of Poiseuille flow? Hint: Look up the curl in cylindrical coordinates. In cylindrical coordinates r, ϕ, z, the vector Laplacian of an azimuthal field is $\nabla^2 \boldsymbol{A} = (\nabla^2 A_\phi - A_\phi/r^2)\hat{\phi}$, while the scalar Laplacian of a function of the radial coordinate alone is $\nabla^2 f(r) = \frac{1}{r}\partial_r(r\partial_r f)$.

5. $\langle \mathbf{5} \rangle$ Suppose we have the no-slip boundary condition on the velocity field $\boldsymbol{v}$ on a fixed solid plane boundary $z = 0$ for flow in the half space $z \geq 0$. What can you say about the vorticity on the boundary?

6. $\langle \mathbf{1+1+2+5} \rangle$ Suppose we have a Cartesian frame for $\mathbb{R}^3$ and make a rotation to a new frame via an orthogonal transformation L. Given a Cartesian tensor with components

$t_{ij\cdots k}$ in the old frame, its components in the new frame are $\tilde{t}_{i'j'\cdots k'} = L_{i'i}L_{j'j}\cdots L_{k'k}t_{ij\cdots k}$ with repeated indices summed from one to three. (a) Express the condition for L to be orthogonal in components. (b) What is $\det L$? (c) Show that the Kronecker symbol has the same components in the new frame: $\tilde{\delta}_{i'j'} = \delta_{i'j'}$ so that it is an isotropic tensor. (d) Show that the Levi-Civita symbol ϵ_{ijk} is also isotropic. Hint: use $\epsilon_{ijk}L_{1i}L_{2j}L_{3k} = \det L$.