Fluid Dynamics, Autumn 2024, CMI

Assignment 10

Due by the beginning of the class on Monday, Nov 11, 2024

Poisson brackets, Functional derivatives, Clebsch variables

1. $\langle \mathbf{2+3+3} \rangle$ Suppose for barotropic flow we postulate the Poisson brackets between functionals F and G of ρ and $\boldsymbol{v}$:
$$\{F, G\} = \int \left[\frac{\boldsymbol{w}}{\rho} \cdot \left(\frac{\delta F}{\delta \boldsymbol{v}} \times \frac{\delta G}{\delta \boldsymbol{v}} \right) + \nabla F_\rho \cdot \frac{\delta G}{\delta \boldsymbol{v}} - \nabla G_\rho \cdot \frac{\delta F}{\delta \boldsymbol{v}} \right] d\boldsymbol{r}. \tag{1}$$
Here, subscripts $F_\rho = \frac{\delta F}{\delta \rho(\boldsymbol{r})}$, etc., denote functional derivatives. Show that these imply the Landau PBs $\{\rho(\boldsymbol{x}), \rho(\boldsymbol{y})\}$, $\{\rho(\boldsymbol{x}), \boldsymbol{v}(\boldsymbol{y})\}$ and $\{v_i(\boldsymbol{x}), v_j(\boldsymbol{y})\}$.

2. $\langle \mathbf{4} \rangle$ Show that the **functional derivative of helicity** $\mathcal{K} = \int_{\mathbb{R}^3} \boldsymbol{v} \cdot \boldsymbol{w} \, d\boldsymbol{r}$ with respect to the velocity field is twice the vorticity field for suitable boundary conditions (to be specified).

3. $\langle \mathbf{5} \rangle$ Show that the Clebsch decomposition of a velocity field $\boldsymbol{v} = \nabla\phi + \frac{\lambda}{\rho}\nabla\mu$ is the sum of curl-free and helicity-free parts.